Puzzle Corner

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Welcome to Puzzle Corner 49 of the *Gazette* of the Australian Mathematical Society. I will start with the new problems and then include a solution of Puzzle Corner 48 in the July issue of the Gazette.

Since this is the Puzzle page, and since, as it says in my signature panel, I invented *Circular Sudoku*, I thought this month I would post two of these puzzles for *Gazette* readers and explain their history. The first ever Circular Sudoku Puzzle appeared in the British newspaper, *The Sunday Telegraph* on 26 June 2005. Since then it has taken on a life of its own: Googling 'Circular Sudoku' leads to millions of hits, for the puzzle has appeared in numerous magazines, newspapers, and on phones and computer games around the world. I have even seen examples on the sides of sandwich wrappers.

Mathematical crazes do truly grip the public imagination from time to time and, being mathematical, become global phenomena. The two most notable in our era were surely Rubik's Cube and Sudoku. Having come up with the idea, my daughter Caroline and I were quick to put out the first book of puzzles in the USA but by Christmas 2005 there was already a hand held computer game on the market.

Left: Circular Sudoku game, packaged. *Right:* The actual game.

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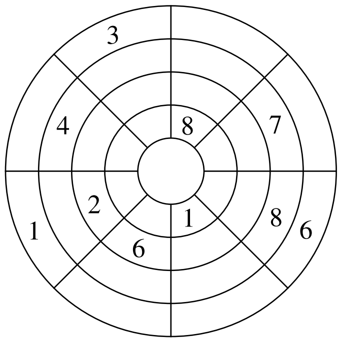
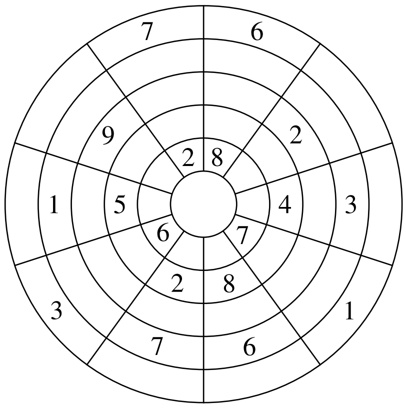
The basic circular sudoku puzzle has rules that can be expressed in one sentence: Every symbol must appear in each of the rings and each pair of adjacent sectors. Armed with just this one instruction, the reader is invited to tackle the two puzzles below.





Unlike regular sudoku, it is quite easy to scale the puzzle up so as to feature more than four rings. For the 4-ring circular sudoku we use the numbers from 1 to 8 while the 5-ring puzzle demands ten symbols and so the numerals $0, 1, \cdots, 9$ work well.

There is a fair bit of underlying mathematics that went into the structure of these puzzles and their many variants. One combinatorial problem associated with regular sudoku was finding the minimum possible number of 'givens' (filled in squares) that could uniquely determine a puzzle. The answer of 17 was eventually proved. One of my masters students solved the corresponding problem for both the four- and five-ring circular sudoku puzzles. In the latter case, the verification was a substantial problem but getting the answer for the four-ring puzzle is not so very difficult. That answer to these questions and the solution to the two puzzles will appear next time.

Solution to Puzzle Corner 48

I will now give my solution to Puzzle Corner 48 which appeared in the July issue of the *Gazette*

1. By first finding $S(\cos x)$, show that $S(\cos(x^2))$ consists of a family of concentric circles together with a family of rectangular hyperbolas.

$$\begin{aligned}
\cos x = \cos y \quad &\Leftrightarrow \quad \cos x - \cos y = 0 \\
&\Leftrightarrow \quad -2\sin\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right) = 0 \\
&\Leftrightarrow \quad x + y = 2k\pi \text{ or } x - y = 2k\pi, \qquad (k \in \mathbb{Z})
\end{aligned}$$

Hence

$$\begin{aligned}(a,b)\in S(\cos(x^2)) \quad&\Leftrightarrow\quad (a^2,b^2)\in S(\cos x)\\ &\Leftrightarrow\quad (a,b)\in\{(x,y)\colon x^2+y^2=2k\pi, k\in\mathbb{Z}\}\\ &\qquad\qquad \cup\{(x,y)\colon x^2-y^2=2k\pi, k\in\mathbb{Z}\},\end{aligned}$$

which represents a family of concentric circles together with a family of rectangular hyperbolas, all centred at the origin.

2. Show that the symmetrizer $S(p(x))$ of a polynomial $p(x)$ of even degree has a contour with an asymptote $x+y=2c$, where c is the mean value of the roots of $p(x)$.

   Clearly we may take $p(x)$ to be monic and so have the form $p(x)=x^{2n}+q(x)$, where $q(x)=a_{2n-1}x^{2n-1}+\cdots+a_0$. There exists a bound N such that if $|x|>N$ then $p(x)>0$ and for any $z\geq N$ there exist unique real numbers $y(z)<0<x(z)$ with $p(x)=p(y)=z$. Note that

$$\begin{aligned}\lim_{z\to\infty}\frac{z}{x^{2n}}=\lim_{z\to\infty}\frac{p(x)}{x^{2n}}=1+\lim_{z\to\infty}\frac{q(x)}{x^{2n}}=1&\\ \Rightarrow\quad \lim_{z\to\infty}\frac{x}{z^{1/2n}}=(1)^{-1/2n}=1.&\end{aligned}$$

   Since $y<0$, we obtain similarly that $\lim_{z\to\infty} y/z^{1/2n}=-1$, so that $\lim_{z\to\infty} x/y=-1$.
   Now in general

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}). \tag{1}$$

   Applying (1) to $(x^2)^n-(y^2)^n$ gives

$$\begin{aligned}&x^{2n}-y^{2n}\\ &\quad=(x+y)(x-y)(x^{2n-2}+x^{2n-4}y^2+\cdots+x^2y^{2n-4}+y^{2n-2})\end{aligned} \tag{2}$$

   Now $p(x)=p(y)\Leftrightarrow x^{2n}-y^{2n}=q(y)-q(x)$. Applying (1) and (2) and dividing this equation in order to make $x+y$ the subject now gives

$$\begin{aligned}&x+y\\ &=-\frac{a_1+a_2(y+x)+\cdots+a_{2n-1}(y^{2n-2}+y^{2n-3}x+\cdots+yx^{2n-3}+x^{2n-2})}{x^{2n-2}+x^{2n-4}y^2+\cdots+x^2y^{2n-4}+y^{2n-2}}.\end{aligned}$$

   Now let $z\to\infty$ so that $x\to\infty$ and $y\to-\infty$. Divide top and bottom by x^{2n-2} and delete terms that approach 0 to obtain

$$\lim_{z\to\infty}(x+y)=-a_{2n-1}\lim_{z\to\infty}\frac{(y/x)^{2n-2}+(y/x)^{2n-3}+\cdots+(y/x)+1}{1+(y/x)^2+(y/x)^4+\cdots+(y/x)^{2n-2}} \tag{3}$$

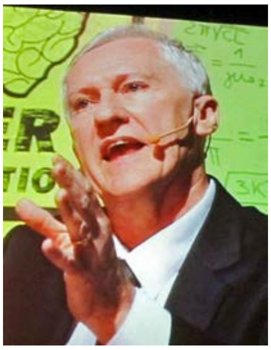
   Since each term in the limit on the right-hand side of (3) approaches ±1 according as the power involved is even or odd, we infer that

$$\lim_{z\to\infty}(x+y)=-\frac{a_{2n-1}}{n} \tag{4}$$

Finally, consider the contour of points (x, y) such that $p(x) = p(y) = z$ where $|x - y|$ is maximum and z ranges of the range of $p(x)$. Writing the limit of (4) as $2c$ we get

$$\lim_{z\to\infty}(x+y) = 2c = -\frac{a_{2n-1}}{n} \quad \Leftrightarrow \quad c = -\frac{a_{2n-1}}{2n},$$

and so c is the mean value of the roots of $p(x)$, counted according to their multiplicity, and the line $x + y = 2c$ is the asymptote of this contour of $S(p(x))$.



Peter Higgins is a Professor of Mathematics at the University of Essex. He is the inventor of Circular Sudoku, a puzzle type that has featured in many newspapers, magazines, books, and computer games all over the world. He has written extensively on the subject of mathematics and won the 2013 Premio Peano Prize in Turin for the best book published about mathematics in Italian in 2012. Originally from Australia, Peter has lived in Colchester, England with his wife and four children since 1990.