Book Reviews

Mathematical Curiosities

Alfred S. Posamentier and Ingmar Lehmann
Prometheus Books, 2014, ISBN 978-1-616-14931-4

The authors are well known as mathematical expositors, and produce about a book a year. This is their 2014 offering, with the subtitle 'A Treasure Trove of Unexpected Entertainments'. It looked like just the sort of book I like: full of interesting tit-bits, discussions, exploration of some of the curious by-ways of mathematics, elementary topics outside the standard high school syllabus — a sort of grab-bag, if you like, of elementary mathematics. There's certainly a great deal in it, but sometimes I felt that there was almost *too* much, and that some topics merited greater depth than they received.

I'm not suggesting that a book for the general reader, or interested student, should have a high seriousness as if it were a research paper, and start with carefully presented definitions, to be followed with lemmas, theorems, corollaries, all with proofs, but I do think that any book of mathematics should endeavour to show some of the wonderful interconnectedness of mathematics, and provide just enough depth to make that possible.

However, it may be that Posamentier and Lehmann are attempting something different, and maybe they deliberately set out to include as many topics as they could. At any rate, I can't fault the book for lacking in breadth!

The contents

The book contains five chapters: Arithmetic Curiosities, Geometric Curiosities, Curious Problems with Curious Solutions, Mean Curiosities, An Unusual World of Fractions. In case some meaning was obscured by its trans-Pacific crossing, I looked up 'curiosity' in the American Heritage Dictionary, to find it means: 'An object that arouses interest, as by being novel or extraordinary', or maybe 'A strange or odd aspect'. I'm putting my money on the first meaning.

So we expect the book then to contain mathematical facts which are 'novel or extraordinary'. Many of them indeed are, but the limited space the authors allow for a topic means that sometimes one would wish for a bit more discussion. Take for example 'The Amazing Number 193,939' on page 40. The authors point out that this is prime, as are indeed five other permutations of its digits. In fact it turns out that there are 23 different permutations of the digits of 193939 which are prime, and the record number of prime permutations for six digit numbers is 123,479, which has 148. A few pages further on the authors discuss Ruth–Aaron numbers, which are consecutive numbers whose sum of prime factors is equal. The smallest such pair is 714 and 715, and they are named for the great baseball player

Babe Ruth, who hit 714 home runs in his career, and Hank Aaron, who beat Ruth's record nearly 40 years later. There are several pages filled with examples of Ruth–Aaron pairs (and triplets) but no discussion about how they can be found, or indeed if they can be generated in some way. The authors missed an opportunity to introduce a result [3] of Carl Pomerance and his students, which he described later [4] as being in a 'short, humorous paper', and which says that if

$$s = 2n+1, \quad p = 8n+5, \quad q = 48n^2+24n-1, \quad r = 48n^2+30n-1$$

are all prime, then $pq+1 = 4sr$ and pq, $4sr$ is a Ruth–Aaron pair. (Incidentally, it was this short humorous paper which was the genesis of the long and fruitful collaboration between Pomerance and Paul Erdős.)

Examples of verbiage in this first chapter consist of the full integer representation of 2^{2^n} up to $n = 9$ (without, however, any mention of Fermat primes), and several pages given over to solutions to the 'four fours' problem: representing every integer from 1 to 100 using only four fours and arithmetic operations. From the slightly blurry look of these digits, I'd say that these were scanned in as an image from another source, which seems lazy.

There are also examples of what look like hasty editing, such as on page 102, when the authors discuss alternating factorial sums:

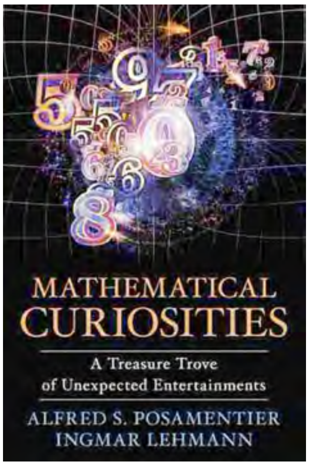
$$n! - (n-1)! + (n-2)! - \cdots - (-1)^n$$

and give the sums for n from 3 to 8, and then say: 'These sums are always prime numbers.' In fact the next sum, with $n = 9$, is composite. What the authors meant to say, and indeed should have said was: 'These first six sums are prime numbers.' They seem to contradict their own assertion by pointing out that further sums are not necessarily prime, which is of course true.

Note that the sequence of integers for which the alternating factorials are (probably) prime is number 001271 in the Online Encyclopædia of Integer Sequences; this wonderful resource gets not a single mention in the book.

The chapter contains a discussion on what the authors refer to as "the most curious 'number' or quantitative concept ∞". The discussion is at times a little unclear, for example when referring to the symbol ∞ as a 'concept'. They say, quite correctly (if a little informally), that 'there are, in fact, orders of magnitude of infinity', but instead of leading into a discussion of Cantor's diagonal argument, they simply show that an infinite set can have a one-one correspondence with a proper subset. This is one place where more discussion is definitely needed: the various uses of infinity in mathematics can be extraordinarily confusing to the learner, and the space here is simply insufficient to clarify possible misunderstandings. I think it is a brave and wonderful thing to include a discussion of ∞; I would have liked to have seen a bit more.

I especially liked the geometry chapter, which contains several sections. One is on Japanese 'Sangaku' or Temple Geometry problems. These were geometric problems posed by anyone, and hung on wooden tablets inside temples during Japan's isolationist Edo period (1603–1868). These problems range from school geometry exercises to highly non-trivial theorems. Possibly the best known is now known as the ‘Japanese theorem’, and states that for any cyclic polygon, the sums of inradii of the triangles in a triangulation is independent of the triangulation. The best account of these problems and their history is by Hidetoshi and Robinson [2]. Here the authors choose only those problems amenable to simple solutions; the solutions are not particularly elegant or unexpected, and contain the usual mixture of algebra and geometry. Some of the diagrams are overly complicated and messy, and there is a confusing mixture of font shapes and sizes. In at least one diagram (2.23) there are so many tiny labels crammed into so small a space that it’s well nigh impossible to work out what’s going on.



There is a section about ‘squaring the square’: that is, subdividing a square into smaller squares no two of which are equal. This is a fabulous problem, and seems to have first been solved by the eminent graph theorist William Tutte and three of his co-students as undergraduates in Cambridge in 1938 [1], although the first published squaring was in 1939 by the German mathematician Roland Sprague. None of these names are mentioned. Instead the authors erroneously claim that the first squaring was achieved in 1964 by J.C. Wilson. But there are some pretty pictures.

There is a nice section on quadrilaterals, which do indeed have a host of interesting elementary theorems. Cyclic quadrilaterals in particular are fascinating beasts. The authors do not distinguish between convex, non-convex, or self-intersecting quadrilaterals.

The third chapter, ‘Curious problems with curious solutions’ is not really either: these are 90 fairly standard elementary mathematical brain teasers, some of which admit to an ‘aha!’ solution; others which simply require some geometry, algebra, or arithmetic.

The fourth chapter on means, is certainly a fascinating topic, and less studied than it should be, but the standard inequality relating the arithmetic, geometric and harmonic means A, G and H is at least well known: $H \leq G \leq A$. The authors unfortunately adopt some very poor notation, writing

$$a \textcircled{A} b, \quad a \textcircled{G} b, \quad a \textcircled{H} b$$

for the arithmetic, geometric and harmonic mean respectively of a and b. The problem with this notation is that it makes these means look like operators, which clearly they are not: $a \textcircled{A} b \textcircled{A} c$ is *not* the arithmetic mean of a, b and c. Moreover, it takes space and adds confusion. Much of the chapter is devoted to geometric proofs of the above inequality, taking in a few other means en route, such as the root mean square, the Heronian mean (which is $(a+\sqrt{ab}+b)/3$ in case you didn’t know), the contraharmonic mean $(a^2+b^2)/(a+b)$ and the centroidal mean $2(a^2+ab+b^2)/(3(a+b))$, (which comes with its own typo as the *controidal mean*) without any discussion of their meaning, contexts, uses or history. Naturally each mean is introduced with the same operator-like notation we saw above. This chapter actually contains some really good mixtures of geometry and algebra, I think let down a little by the notation and diagrams. The chapter finishes with the claim that mean inequalities 'is one of the neglected curiosities in mathematics worthy of our attention', which is unfortunately true, at least as far as elementary teaching goes.

The last chapter is mostly given over to discussing Leibniz' harmonic triangle, which consists of the fractions $1/n$ down each side, and for every triangular group

$$\begin{array}{ccc} & a & \\ b & & c \end{array}$$

then $a = b+c$. So every fraction is equal to the sum of the two fractions immediately below it:

$$\begin{array}{c}
\frac{1}{1} \\
\frac{1}{2} \quad \frac{1}{2} \\
\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3} \\
\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4} \\
\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5} \\
\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6} \\
\frac{1}{7} \quad \frac{1}{42} \quad \frac{1}{105} \quad \frac{1}{140} \quad \frac{1}{105} \quad \frac{1}{42} \quad \frac{1}{7} \\
\frac{1}{8} \quad \frac{1}{56} \quad \frac{1}{168} \quad \frac{1}{280} \quad \frac{1}{280} \quad \frac{1}{168} \quad \frac{1}{56} \quad \frac{1}{8}
\end{array}$$

The authors miss an opportunity of showing that every element in the triangle must be a unit fraction (which is not obvious from the definition), although they do show that the elements in the 'second oblique' are the fractions

$$\frac{1}{n(n+1)}.$$

Having discussed triangles, they can't resist a look at Pascal's triangle, but miss out on the basic relationship between the two triangles: if Leibniz' triangle is indexed from 1, then

$$L(m,n) = \frac{1}{m\binom{m-1}{n-1}}$$

Thus the denominators in Leibniz' triangle can be obtained by multiplying the nth row in Pascal's triangle by $n+1$. This is hinted at, but not stated as a general result.

This chapter, and the book, finishes with a discussion of the mediant of two fractions, defined as

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

The authors show that if all values are positive, then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

and provide a few proofs, including a nice geometric one. Then they segue into Farey sequences and Ford circles.

Conclusion

I think that this book contains plenty of absolutely terrific stuff, unfortunately let down by lack of explanations, some poor choices of notation, and some unclear diagrams. That being said, there are also many places where the explanation (including the diagrams) is appropriate for the material. There are so many ways in which material from different parts of the book could be interrelated — and in my own experience as a teacher, showing that two apparently dissimilar topics are in fact deeply related provides a tremendous interest to students — but unfortunately there are none in this book.

The authors' love of mathematics, and appreciation for its breadth and history shines through on every page — for me this was one of its most attractive aspects.

The book could be considered, as I said in my introduction, as a sort of 'grab-bag' of elementary mathematics; any student would almost certainly find something to spark her or his interest in the book, and then could follow it up with greater depth either with the bibliographies at the end of each chapter, or online.

References

[1] Gardner, M. (1961). Squaring the square. In *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* Simon and Schuster, New York, pp. 186–209
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[3] Nelson, C., Penney, D. E. and Pomerance, C. (1974). 714 and 715. *J. Recreational Math.* **7,** 87–89.
[4] Pomerance, C. (2002). Ruth–Aaron numbers revisited. In *Paul Erdős and his Mathematics.* Bolyai Soc. Math. Stud. 11, pp. 567–579.

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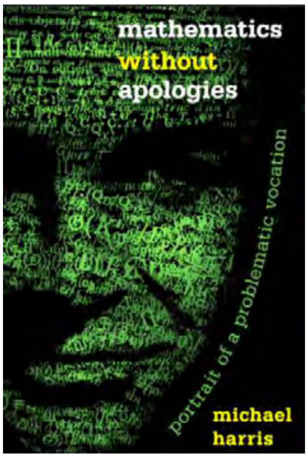
Mathematics Without Apologies

Michael Harris
Princeton University Press, 2015, ISBN 978-0-691-15423-7

This book was not the book I expected. It was at turns frustrating and cause for reflection on our field. While Harris openly says that this is not a scholarly work, and intended for the interested bystander, it is no *Fermat's Last Theorem* [3]. The

24-page bibliography and 68 pages of endnotes[1] lend an air of serious study. Others I know have merely sampled from the book, picking out the main points; perhaps this is sufficient until one wants to spend the effort to plough through the constant references to continental philosophers and literary theorists. This book reads like a work in sociology, but with a familiarity with mathematics, mathematicians and the current state of play one would be surprised to find in a sociologist.



That said, this book has grown on me since I put it down; it made me go back and read Hardy's *Apology* [1], for the sake of comparison, and I think there are some points from Hardy that are little referenced, and which are pertinent to our time, and Harris's update. Many know of the famous parts from Hardy: the passages about Ramanujan (for instance the 1729 story), the references to the young man's (sic) game, the exultation in the uselessness of number theory (now undone by modern cryptography) and so on. Hardy attempts to answer the question 'why do mathematics?', and then immediately confesses that, while posing several general potential answers, he is answering the question 'why did *I* do mathematics?'.

Hardy does, however, make a general point that we can no longer take for granted, and to which one can view Harris's book as an extended riposte:

> I may be told that mathematics needs [no apology], since there are now few studies more generally recognized, for good reasons or bad, as profitable and praiseworthy...The public does not need to be convinced that there is something in mathematics.

This viewpoint is, at least in the Anglosphere, terribly quaint, and almost laughable now; declining numbers of students taking mathematics tell us what 'the public' think of our field. Hardy then goes on to say how poetry is more valuable than cricket, to highlight why Bradman should play cricket rather than write poetry: it is the former that he does really well. The explicit implication is that mathematicians should stick to what they can do well, despite its comparative worth to other endeavours. Mathematics today is seen more as the poetry than the cricket by some. By way of illustration, Hardy mentions finance and the legal profession as occupations people might do because they lack facility in other areas. Harris, unintentionally, mirrors this, with discussion of mathematics and mathematicians' role in the financial crisis of the past decade.

So what is Harris's non-apology, as given away by the title? Namely this: that we do mathematics because we like to, that it has a sort of freedom of play, and,

[1]This reviewer bravely tried to read all the endnotes in context, but cannot claim to have done so successfully.

more implicitly, because it brings a form of prestige, for which he uses the term 'charisma'. This charisma is evident in people such as Cedric Villani, the Fields medallist who has shot to national fame in France, but also, justly, holds high position in mathematical circles; Robert Langlands, who as Edward Frenkel likes to emphasise, has Einstein's old office in Princeton; and our own Cheryl Praeger, recognised for her research but also trailblazing for women mathematicians in Australia.

Harris illustrates the, or at least *a*, path to charisma by telling what seems like his own story, but which he has since on his blog [2] declaimed as being fictionalised.[2] The 'Michael Harris' in the book is not Michael Harris the author, or so he says. At the very least, the path of finding oneself integrated into the Langlands programme and proving significant parts of it, is not one that most mathematicians can identify with. Finding oneself fêted and published by the 'great journals' is a distant dream for those that make up the rank and file of mathematicians.

The 'relaxed field' that Harris discusses — while perhaps true while we are in the business of proving theorems, playing with our mathematics, doing that which identifies us as mathematicians — is perhaps not the same for us as for those with charisma. The demands of day-to-day work, the administration, the teaching, the grant or job applications and so on, are not what one thinks of when reading the Cantor quote 'The essence of mathematics is in its freedom'. But we can perhaps see a lesser shadow of it, and certainly for this reviewer it played a big rôle in the decision to do mathematics long-term. Harris, however, doesn't shirk examining the everyday activities mentioned above. He makes a careful study of what *being* a mathematician is like. Reading this is a bit like discovering one has been speaking prose. Being written for those outside the system of mathematicians (and journals, and grants, and ...) it is an in-depth look at what we do, and generally how it works. The Gowerses, Taos and Villanis of the world, and presumably the Harrises (fictional or otherwise), still have *some* admin to do, still need to pass the editors, the gatekeepers of our literature, and so this section rings very true.

To give a bit more of an idea of what mathematicians do in terms of mathematics, in particular what he himself does, Harris indulges in a fictionalised account of an extended discussion between an anonymous Number Theorist and a Performing Artist, scattered in several chapters labelled by Greek letters. Each of these dialogues is preceded by what one might recognise as conventional 'popular mathematics book' material, to get readers up to speed. One can read the Greek-lettered chapters on their own as short, an almost independent book; likewise with the complement of the book. But the mathematics is not restrained to these chapters. Liberal discussion of contemporary mathematics in a more impressionist style is scattered throughout. In fact, I was pleasantly suprised to find Harris at home in

[2]Harris's blog, with the same name as the book, is a must for readers of the book. One can find reviews by others, together with takedowns by Harris of those he feels miss the point. Apparently it is common for reviewers to not get the point of the book, a fact to which this reviewer can only gesture mutely; he fully expects to join their ranks. But the existence of the blog allows Harris to explain himself more fully outside the constraints of the linear narrative of the book, and to further illustrate his points with current events, or sources not available at the time.

the digital humanities: he cites blogs (e.g. Terry Tao's *What's New*, the *n-Category Café*, *mathbabe*), MathOverflow, YouTube videos, Wikipedia articles, databases of rap lyrics and so on. Given the amount of mathematics that happens online today, and mathematics that is recorded online that would have been otherwise lost (for instance, Vladimir Voevodsky's contentious talk *What if current foundations of mathematics are inconsistent?*, given at the Institute for Advanced Study) it is only appropriate to include that in a study of what we do and why.

Which brings us back, finally, to what is Harris's main point: we do mathematics because we like to. Very few could honestly claim to do mathematics, or at least *pure* mathematics, entirely out of altruism to better humanity, or because of potential technological payoffs that may take a century to eventuate (the 'golden goose' argument, as Harris puts it). In this one sees why the book is titled *Without Apologies*. Hardy makes a similar point: 'Any genuine mathematician must feel it is not on these crude achievements that the real case for mathematics rests', where by 'crude achievements' he means practical applications such as 'bridges and steam engines and dynamos'. Villani seems to be fond of quoting, in talks and interviews, his countryman Weil on the pleasure of doing having mathematical ideas, which can last for 'hours at a time, even for days', and one wonders whether this is the ultimate reason for many for pursuing mathematics. It is worth contrasting Villani's recent book [4] with Harris's. It takes the approach of *showing*, rather than *telling* (like Harris), what mathematicians do, and his own enjoyment of it. The inclusion of emails between Villani and his collaborator quoted verbatim, complete with pseudo-LaTeX equations, is perhaps off-putting to the general reader (mathematicians may very much love it!), but then so might Harris's continental philosophy and sociology, his deconstruction of Pynchon novels in mathematical themes.

All in all, Harris's book is something people interested in the whys and wherefores of doing mathematics and being a mathematician should read. If one has curious friends or relatives of a literary bent, or versed in some form of humanities, then this may be the book for them. Despite being listed as general interest, some facility with (or willingness to look up) literature and philosophy spanning several languages is recommended.

References

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[2] https://mathematicswithoutapologies.wordpress.com.
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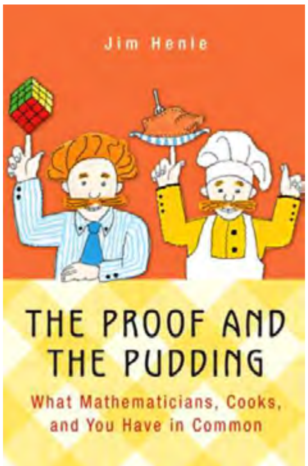
The Proof and the Pudding: What Mathematicians, Cooks and You Have in Common

Jim Henle
Princeton University Press, 2015, ISBN 978-1-400-86568-0

The Proof and the Pudding is not a book about mathematics *for* cooking.[1] On the back jacket of this book, a reviewer has been quoted as saying '[Henle] challenges mathematicians to be chefs and chefs to be mathematicians'. To me, this is not at all the message of the book.

The author compares the ways of thinking and acting that are common between his twin loves, mathematics and gastronomy. I'm not sure where bookshops put this book. The publishing data gives it a classification among mathematics books (Dewey Decimal 510), though it could equally well be found on the 640 shelves (home economics); my own preference would be to house it with motivational or study skills.



The practical and material aspects of gastronomy and mathematics are discussed and dismissed in less than a page (on page 110!). The author has a loftier purpose. This book is actually — if you read between the recipes and the maths puzzles and the excellent diagrams that illustrate them — about the need to be at times humble and self-doubting and at other times arrogant or confident, to experiment and learn by trial and error, to refine one's skills and products, to seek what is aesthetically pleasing, to be economic of effort, to persist, and to find enjoyment in one's labours as well as the fruit of them. Henle has distilled these ideas from his own experiences of cooking for his family and friends, and of mathematical collaboration and research. It is funny in many places, and inspiring. If you have ever served up a failed mousse or thrown away a couple of days' calculations, the stories in this book will resonate.

This is a personal reflection, written in an informal and individual tone, and you might not enjoy the style as much as I did. You may find some of the parallels a little forced and you might not like the sound of Shamburgers or blue pizza. You may not agree that mathematics has so much in common with all other fields of human endeavour (if that implies it is not quite so special). But you will find some cute little proofs and problems to illustrate your lectures, some Nim-like games to play and a very sophisticated card-trick. When we (or our students, perhaps our

[1] However, I did find a very useful fact (for non-Americans) on page 58: $1\frac{1}{3}$ sticks of butter $= \frac{2}{3}$ cups. I have often pondered this question. It seems you can't just pick your own stick.

graduate students) aren't as successful or creative in our mathematics as we are in our cooking/carpentry/figure-skating/gardening/photography/[insert your own hobby here], Henle reminds us

> It's just a question of desire. The same attitudes, the same mental approaches, the same problem-solving skills propel you forward. (p. 149)

Now, if I can work out how many punnets of blueberries make up a pint, I might be able to try the three-ingredient pudding recipe.

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Introduction to Nonsmooth Optimization: Theory, Practice and Software

Adil Bagirov, Napsu Karmitsa, Marko M. Mäkelä
Springer, 2014, ISBN 978-3-319-08113-7

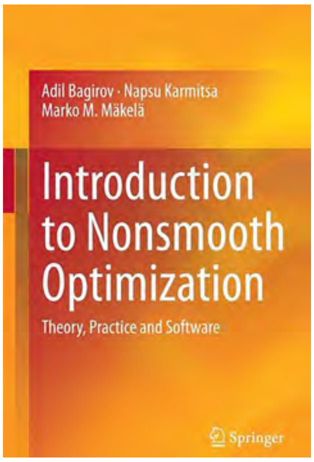
This book is a welcome and much needed contribution to the field of Nonsmooth Optimization. In this self-contained text, the reader will find (i) the main theoretical tools for studying convex and nonconvex problems, (ii) concrete examples in which these tools are used, and (iii) a comprehensive guide to the available software to implement the solution techniques.

A nonsmooth optimization problem consists of minimising a function which may not be differentiable (in the classical sense) at the solution. This situation is relevant because it provides a more realistic representation of a wide variety of problems. Moreover, several new tools and techniques have recently been developed for these problems, and practitioners would greatly benefit from understanding when and how they can be implemented. This book addresses this need in a superb way. It is self-contained, it is very easy to read, and all the material has been arranged in three very well-organized parts.

Namely, in a single text, the reader has access to the three different and supplementary aspects of nonsmooth optimization listed above in (i)–(iii). Covering these three aspects in a comprehensive way is, as far as I know, a distinctive characteristic that makes this book unique among the currently available texts, and particularly appealing to both theorists and practitioners.

Readers are only required to have some basic background in linear algebra, multidimensional calculus and elementary real analysis. At the end of almost every chapter exercises are given and the contents are summarised. As a result, parts of this book can be used as a text for advanced (or masters/honours) students in mathematics or engineering degrees. Rigorous mathematical proofs are given for most of the statements.

Part I collects the relevant theoretical tools on which nonsmooth optimization relies. Chapter 1 includes facts and definitions about locally Lipschitz functions. This type of function is to have a key role in subsequent chapters. Chapter 2 is devoted to convex analysis. The first part of the chapter has a geometric approach, focusing on projections, separation properties and convex cones. Theoretical properties of convex functions and their subdifferentials are studied in detail, and many illustrations and examples are presented. The epsilon-subdifferential and its connections with the classical subdifferential are clearly presented and motivated by their use in nonsmooth optimization methods. It is captivating to see in Section 2.3 how all the analytical tools are connected with geometrical objects, giving the reader the intuition to understand and enjoy better the theory.



Chapter 3 in Part I is devoted to nonconvex analysis, and it is here that the convex theory is extended to functions which are locally Lipschitz continuous. Since the latter may not be differentiable in the classical sense, the authors start by recalling the Clarke subdifferential and the generalized directional derivatives of locally Lipschitz functions, which will play the role of the derivatives for this more general family of functions. This is followed by the calculus rules that apply to these objects. Several illustrative examples are given to show the subtleties and the different aspects of the definitions, especially to point out that nonsmoothness can exist even for functions that are everywhere differentiable (Example 3.1). The role of the epsilon subdifferential in the convex case is now played by the Goldstein epsilon-subdifferential. The Jacobian of a vector-valued function is extended to locally Lipschitz continuous functions and subdifferential calculus is given in great detail. This includes the extensions of the chain rule, the mean value theorem, etc., to these concepts, accompanied by numerical examples. The geometrical aspects of nonsmooth problems, and their analogies with the convex case, are presented through the (Clarke) tangent and normal cones in Section 3.3. Other kinds of subdifferentials are recalled in Section 3.4, where the reader can also find the relationships between these concepts and the Clarke subdifferential. Chapter 4 presents the optimality conditions for nonsmooth optimization, both for the unconstrained and the constrained case. Very clear proofs, geometrical results, and illustrative examples give further motivations and captivate the attention of the reader. Special attention is given to the classical model using inequality constraints. Results which are analogous to Fritz John and Karush Kuhn Tucker conditions are presented and illustrated for locally Lipschitz functions in Section 4.3.

Chapter 5 considers problems that enjoy additional structure, and explains how this additional structure can be exploited. The functions of the problem are assumed here to be pseudo- or quasi-convex, which are two important cases in which the convexity assumption is relaxed. The authors show how these assumptions can be used to derive optimality conditions. The main tool for deriving the results are the Clarke subdifferential and the generalized directional derivatives. Relaxed optimality conditions are given for the unconstrained and constrained problems, as well as for the model using inequality constraints. In all cases, the functions involved are either pseudo- or quasi-convex. In particular, Karush Kuhn Tucker conditions are recovered for the model with inequality constraints, in which the objective function is pseudo convex and the constraints are quasi-convex.

Chapter 6 focuses on approximating subdifferentials by means of continuous set-valued functions. In this context, it studies a concrete approximation of the subdifferential, called discrete gradient (introduced in 1999 by Adil Bagirov). The advantage of the discrete gradient is that it allows you to compute search directions by using only functional values. The concept of continuous approximation allows you to obtain in Theorem 6.8 an optimality condition in terms of discrete gradients. Section 6.3 considers piecewise partially separable problems, and shows how to compute the discrete gradients for these functions. These topics link with Chapter 15 in Part III, where it is shown (through two different methods) how the discrete gradients can be implemented. At the end of Part I, comprehensive historical references are given for all the concepts given in Chapters 1–6.

Part II of the book considers practical formulations of nonsmooth optimization problems of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \end{array}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is only assumed to be locally Lipschitz continuous over the set S. The authors classify the source of nonsmoothness as: physical, technological, methodological, and numerical. This part of the book focuses on all aspects of nonsmooth problems, going from the theoretical formulation of a comprehensive list of problems in Chapter 7, to a systematic list of nonsmooth test problems of various types in Chapter 9.

Chapter 7 describes in detail several important real-life nonsmooth problems, arising from computational chemistry and biology, data mining, optimal control, image denoising, and economics. All these models are explained in detail, which is very useful for mathematicians wanting to look into interdisciplinary applications. Of particular interest is Section 7.2 on Data Analysis, which presents a generalization of classical convex separation to the so-called polyhedral separation, whereby disjoint nonconvex sets are separated using piecewise linear functions.

We mentioned above that the authors classify nonsmoothness in four distinctive classes, and one of them is methodological nonsmoothness. Typical examples of this type of problems are Lagrangian relaxation, dual reformulation, and exact penalty functions. In these cases, a possibly smooth problem is reformulated as a nonsmooth one, where the loss in terms of differentiability is compensated by a gain in structure (and hence resulting in a methodological gain). This is the focus of Chapter 8, which describes the exact penalty formulation, integer programming with Lagrange relaxation, and the maximum eigenvalue problem.

The last chapter of Part II, Chapter 9, consists of an extensive and carefully classified list of test problems for nonsmooth optimization. All problems in this chapter are openly available and have been used to test, develop or compare nonsmooth software. Problems of most sizes and types can be found in this list, so this chapter will be an essential reference for those who want to understand, and/or compare the behaviour of methods for nonsmooth optimization. All the references to the problems and the historical notes are given at the end of this Part II.

Part III describes the main solution techniques used for solving nonsmooth problems. Subgradient-, cutting plane-, bundle-, and gradient sampling methods are discussed in Chapters 10–13. In Chapter 14 some combinations of the previous techniques are presented, including variable metric bundle, and limited memory bundle methods. The discrete gradient and some of its variants are presented in detail in Chapter 15.

Each technique given in Chapters 10–15 is explained theoretically, and the available convergence results are given. Moreover, pseudo-codes are given for all the methods.

All methods in Chapters 10–15 refer to the unconstrained problem. Chapter 16, however, explains two ways in which constraints can be incorporated into the objective, in such a way that the problem becomes unconstrained. The first one recalls the exact penalty approach given in Chapter 8, and gives a pseudocode for implementing it. The second one is the linearisation approach. The description of the latter approach is given for the convex case.

Finally, Chapter 17 compares different implementations of nonsmooth optimization methods. The comparisons here use a very large number of test problems, and the methods considered in Chapters 10, 12, 14 and 15 are included in the tests. Chapter 17 is designed to give insight into which method should be used for a given problem. A comprehensive list and description of available nonsmooth solvers and the corresponding references are given in Section 17.1. In Section 17.2, these methods are tested and compared using representative subsets of the test problems given in Chapter 9. These comparisons are made for problems of different sizes and types (convex and nonconvex). Convergence rates and the iteration paths are experimentally studied as well.

In summary, this book is an extremely valuable reference that collects and relates all the different facets of nonsmooth optimization, in a way that is comprehensive, captivating, and easy to follow.

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