Technical Papers

Configuration spaces in topology and geometry

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The nth *configuration space*, $\mathrm{Conf}_n(X)$, of a topological space X, is the space of n distinct points in X. In formulas,

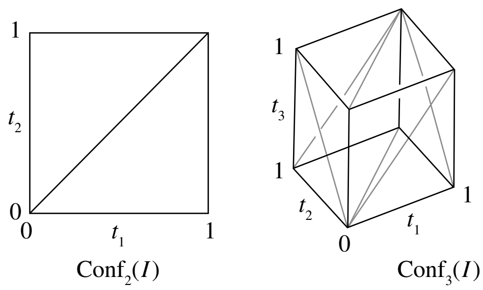
$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \mid x_i \neq x_j \text{ if } i \neq j\}.$$

This is often called the *ordered* configuration space. There is a natural action of the symmetric group S_n on $\mathrm{Conf}_n(X)$ which reorders the indices of the n-tuple; the quotient $\mathrm{Conf}_n(X)/S_n$ by this action is therefore the *unordered* configuration space.

This family of spaces has been studied from many points of view. For instance, in gravitation, $\mathrm{Conf}_n(\mathbb{R}^3)$ is the natural home for the n body problem (see [6] for the interesting history of this topic). Configurations of points in $\mathbb{R}^3$ that are required to conform to a given geometry (say, that of a robot's arm) are employed in robotics and motion planning[1]; see, for example, [10]. In this note, we will discuss the appearance of these spaces in homotopy theory and algebraic geometry.

Example 1. Let I denote the open interval $I = (0, 1)$, and examine $\mathrm{Conf}_n(I)/S_n$. Up to reordering, n distinct points in I are given by an increasing sequence $0 < t_1 < t_2 < \cdots < t_n < 1$ of real numbers. The collection of such n-tuples $(t_1, \ldots, t_n)$ is called the *open n-dimensional simplex*. For instance, when $n = 1, 2, 3$, these are easily seen to be I, an open triangle, and an open tetrahedron.

So $\mathrm{Conf}_n(I)/S_n$ is an open simplex; $\mathrm{Conf}_n(I)$ is simply a disjoint union of $n!$ copies of this, since there are precisely $n!$ different reorderings of the t_i.



Invited technical paper, communicated by Mathai Varghese.

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[1]Conrad Shawcross's kinetic sculpture 'Loop System Quintet', on display at MONA in Hobart, Tasmania, gives a wonderful realisation of such a configuration space.

Configuration spaces for different spaces X and X' are often related if X' is obtained from X by removing a finite set of points.

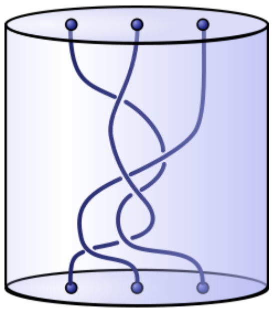
Example 2. Let S^1 denote the unit circle in $\mathbb{C}$ and $I = (0, 1)$. There is a homeomorphism $f\colon S^1 \times \mathrm{Conf}_n(I) \to \mathrm{Conf}_{n+1}(S^1)$ given by

$$f(z, t_1, \ldots, t_n) = (z, z\mathrm{e}^{2\pi i t_1}, \ldots, z\mathrm{e}^{2\pi i t_n}).$$

Thus $\mathrm{Conf}_{n+1}(S^1)$ is simply the product of a circle with a union of $n!$ open simplices. The action of S_{n+1} is somewhat harder to visualise in this description.

We are interested in the topology of configuration spaces; at first blush, let us examine the fundamental group $\pi_1(\mathrm{Conf}_n(X))$.

Example 3. First, note that a *path* in $\mathrm{Conf}_n(X)$ consists of an n-tuple of paths of distinct points in X; this is a *loop* if it starts and ends at the same configuration. Replacing such a family of paths with their graphs, we see that a loop in $\mathrm{Conf}_n(X)$ can be regarded as an n-tuple of nonintersecting arcs in $X \times I$ that begin and end at the same collection of n points. This is called an *n-strand braid in $X \times I$*; when $X = \mathbb{R}^2$ this recovers the usual notion of braids in 3-space, and indeed the fundamental group $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2)) =: P\beta_n$ is Artin's (pure) braid group [1].



In fact, to the eyes of homotopy theory, this is a complete description of $\mathrm{Conf}_n(\mathbb{R}^2)$. Through an inductive procedure similar to that of Example 2, one can show that all of the higher homotopy groups of $\mathrm{Conf}_n(\mathbb{R}^2)$ vanish [9], and so all homotopy theoretic questions about $\mathrm{Conf}_n(\mathbb{R}^2)$ may be reduced to algebraic questions about the braid group.

Function spaces. One application of configuration spaces is to the study of *spaces of functions*. Here, if X and Y are topological spaces, we will write $\mathrm{Map}(X, Y)$ for the topological space of all continuous functions from X to Y, equipped with the compact-open topology. If X and Y have basepoints, we will write $\mathrm{Map}^*(X, Y)$ for the subspace of maps that carry the basepoint of X to that of Y. These are very large (usually infinite dimensional) spaces.

Finite dimensional approximations to $\mathrm{Map}(X, Y)$ can sometimes be given using configuration spaces. For instance, there is an 'electrostatic map' [11]

$$e\colon\ \mathrm{Conf}_n(\mathbb{R}^k)/S_n \to \mathrm{Map}^*_n(S^k, S^k).$$

(Here Map_n indicates that the degree of these maps is n.) One may assign an electric charge to each element of a configuration $\underline{x} := (x_1, \ldots, x_n)$ in $\mathbb{R}^k$. The associated electric field (with poles at the x_i) may be regarded as a function $e(\underline{x})$ from $\mathbb{R}^k$ to $S^k = \mathbb{R}^k \cup \{\infty\}$ which extends naturally over infinity. It is a consequence of the work of many authors (for example, [13], [2], [11], [14]) that configuration spaces 'see' large parts of the topology of function spaces, for example:

Theorem 1. *The induced map* $e_*\colon H_p(\mathrm{Conf}_n(\mathbb{R}^k)/S_n) \to H_p(\mathrm{Map}^*_n(S^k, S^k))$ *is an isomorphism in homology in dimensions* $p \leq n/2$.

When $k = 1$, we note that by Example 1, the domain of e is a simplex, and thus contractible. The codomain $\mathrm{Map}^*_n(S^1, S^1)$ is also contractible to a standard degree n function $g_n(z) = z^n$. That is, every degree n map $f\colon S^1 \to S^1$ may be lifted over $\exp\colon \mathbb{R} \to S^1$ to a map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ (carrying $\mathbb{Z}$ to $n\mathbb{Z}$), where it may be deformed linearly to $\widetilde{g}_n(x) = nx$. Theorem 1 is, in this case, a triviality.

In contrast, in dimension 2, Theorem 1 suggests that the homotopy theory of S^2 — that is, the study of the (rather mysterious) groups $\pi_{2+m}(S^2) \cong \pi_m(\mathrm{Map}^*(S^2, S^2))$ — may be approached using braid groups. This idea has been realised in the remarkable work of Fred Cohen and Jie Wu [5], who relate those homotopy groups to the descending central series in $P\beta_n$.

Moduli spaces. Configuration spaces are also closely related to *moduli spaces*. The moduli space of n points on the Riemann sphere[2] is the set of n-tuples in S^2, up to conformal automorphisms of S^2, that is:

$$\mathcal{M}_{0,n} := \mathrm{Conf}_n(S^2)/\,\mathrm{Aut}(S^2).$$

Here $\mathrm{Aut}(S^2) = PSL_2(\mathbb{C})$ acts on $S^2 = \mathbb{C} \cup \{\infty\}$ (and hence n-tuples in S^2) through Möbius transformations. Noting that for any $x_1 \in S^2$, there is a Möbius transformation T carrying x_1 to ∞, we have a homeomorphism

$$\mathcal{M}_{0,n} \cong \mathrm{Conf}_{n-1}(\mathbb{C})/\,\mathrm{Aut}(\mathbb{C}) = \mathrm{Conf}_{n-1}(\mathbb{C})/\,\mathrm{Aff}(\mathbb{C})$$

obtained by applying T to a configuration, and dropping ∞ from the configuration. Lastly, the group $\mathrm{Aff}(\mathbb{C}) = \mathbb{C}^\times \ltimes \mathbb{C}$ acts by affine transformations on the plane; we note that it is homotopy equivalent to the subgroup S^1 of rotations.

Combining this with the approximation from the electrostatic map, we see that the homology of $\mathcal{M}_{0,n}/S_n$ is isomorphic in a range to that of $\mathrm{Map}^*_{n-1}(S^2, S^2)/S^1$, where the circle group acts on functions by rotating the codomain. That homology is quite complex indeed (see, for example, [4]), but is in fact entirely torsion[3]! This gives the surprising result:

$$H_p(\mathcal{M}_{0,n}/S_n; \mathbb{Q}) = 0, \text{ if } p > 0. \tag{1.1}$$

Hurwitz spaces. We would like to use these sorts of methods to compute algebro-topological invariants of other families of moduli spaces. Configuration space techniques are particularly well adapted to moduli spaces of surfaces equipped with structures that degenerate at a finite set of points. A good example are *Hurwitz spaces* — moduli spaces of branched covers of Riemann surfaces.

Let G be a finite group and $c \subseteq G$ a union of conjugacy classes. A (G, c)-*branched cover of the sphere* is a Riemann surface Σ equipped with a map $p\colon \Sigma \to S^2$ which, away from a set of n points in S^2, is an analytic, regular covering space with Galois group G. Furthermore, we insist that the monodromy of the cover around the

[2]For surfaces of positive genus, the moduli space is slightly more complicated, and involves the space of constant curvature metrics on the surface.
[3]This is related to the fact that $\pi_k(S^2)$ is torsion except when $k = 2$ or 3.

branch points lie in c. The moduli space of such maps p will be denoted $\mathrm{Hur}^c_{G,n}$; this is the set of all such maps up to conformal automorphisms of S^2.

The topology of $\mathrm{Hur}^c_{G,n}$ is not difficult to describe: the forgetful map $\Phi : \mathrm{Hur}^c_{G,n} \to \mathcal{M}_{0,n}/S_n$ which carries p to its branch locus is a covering space. Now, a branched cover of S^2 may be reconstructed from its monodromy around the branch points. Thus the fibre of Φ over a point $[x_1, \ldots, x_n] \in \mathcal{M}_{0,n}/S_n$ is the set of possible values for that monodromy. This is is the set

$$S = \{(g_1, \ldots, g_n) \in c^{\times n} \mid g_1 \cdots g_n = 1\}.$$

(The product of the local monodromies must be 1 in order for the cover to extend from a neighborhood of the branch locus over the rest of the sphere.) An explicit formula for the action of the (spherical) braid group $\pi_1(\mathcal{M}_{0,n}/S_n) = \beta_n/Z(\beta_n)$ on S associated to this cover is given by

$$\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n),$$

where σ_i is the braid that swaps the ith and $i+1$st strands.

While this is a beautiful description of $\mathrm{Hur}^c_{G,n}$, it is not one that immediately lends itself to computations of homology groups. In joint work with Jordan Ellenberg and Akshay Venkatesh [7, 8], we have adapted the classical techniques to this setting, and computed the homology of $\mathrm{Hur}^c_{G,n}$ for large values of n. As in Theorem 1, it is given in terms of a space $\mathrm{Map}^*_n(S^2, A(G,c))$ of functions from S^2 (that is, the base of the branched covering) into a certain classifying space $A(G,c)$ for branched covers. The topology of the space $A(G,c)$ is quite complicated. However, to the eyes of rational homology, it is nearly indistinguishable from S^2. A sample result, along the lines of equation (1.1), is:

Theorem 2. *Let G be a group of order $2p$, with p odd; let c be the conjugacy class of involutions and A the unique normal subgroup of G of order p. Then there exists a constant*[4] *$\alpha > 0$ so that each component of $\mathrm{Hur}^c_{G,n}$ has vanishing positive Betti numbers in dimensions less than αn.*

Using techniques of étale cohomology, this has, as a corollary, a function field analog of the Cohen-Lenstra heuristics [3] on the distribution of class groups of imaginary quadratic number fields. For function fields, the corresponding heuristics concern the statistics of hyperelliptic curves (degree 2 ramified covers of $\mathbb{P}^1$) over $\mathbb{F}_q$ with prescribed class group (or Picard group). Now, the Hurwitz space in question parameterises branched G covers of $\mathbb{P}^1$ with ramification occurring away from $A < G$; if A is abelian, this is the same as a hyperelliptic curve C equipped with a surjection $Pic(C) \to A$. Thus $\mathrm{Hur}^c_{G,n}$ is the home of the counting problem that the Cohen-Lenstra heuristics present; understanding its cohomology leads to a proof of the heuristics.

[4]Here α depends upon G, and is much smaller than the constant $\frac{1}{2}$ of Theorem 1.

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