Calculus: A Marxist approach

Charles Fahey*, C.T. Lenard** ***, T.M. Mills** **** and Lex Milne** *****

Introduction

Readers of the *Gazette* may be surprised to learn that Karl Marx was interested in calculus. After reading Marx's notes on calculus, Friedrich Engels wrote to Marx as follows on 10 August 1881 [8, p. xxvii].

> Yesterday I found the courage at last to study your mathematical manuscripts even without reference books, and I was pleased to find that I did not need them. I compliment you on your work. The thing is as clear as daylight, so that we cannot wonder enough at the way the mathematicians insist on mystifying it. But this comes from the one-sided way these gentlemen think. To put $dy/dx = 0/0$, firmly and point-blank, does not enter their skulls.

What did Marx write about calculus in his notes? What were his sources? Why was Marx interested in calculus? In this paper, we will endeavour to answer these questions. In doing so, we see that Marx developed his own views on the foundations of differential calculus independently of the ideas of Cauchy and other key figures in 19th century mathematics who wrestled with these issues.

Marx's notes have been translated and are readily available in a collection of papers [8] which is an English translation of Russian translations prepared by S.A. Yanovskaya. The core of this collection consists of several manuscripts in which Marx recorded his thoughts and ideas as he learnt about calculus. Also included are early drafts of some of the manuscripts, together with commentaries by Yanovskaya, E. Kol'man, and C. Smith.

Marx the philosopher and historian

Marx was no dilettante, nor was he a mathematician. He was though, an economist, historian, and philosopher, and it was with the tools of his philosophy that he tackled calculus. To better understand his mathematical writings, we need to

Received 23 February 2009; accepted for publication 30 June 2009.
*School of Historical and European Studies, La Trobe University, VIC 3086.
E-mail: c.fahey@latrobe.edu.au
**Department of Mathematics and Statistics, La Trobe University, PO Box 199, Bendigo, VIC 3552.
***E-mail: c.lenard@latrobe.edu.au
****E-mail: t.mills@latrobe.edu.au
*****E-mail: l.milne@latrobe.edu.au

understand Marx's philosophy. For many people, Marx is almost synonymous with 'dialectical materialism'. Bertrand Russell [12, p. 749] writes,

> His sort [of materialism], which, under Hegelian influence, he called 'dialectical', ... was more akin to what is now called instrumentalism. ... In Marx's view, all sensation or perception is an interaction between subject and object; the bare object, apart from the activity of the percipient, is a mere raw material, which is transformed in the process of becoming known. Knowledge in the old sense of passive contemplation is an unreal abstraction; the process that really takes place is one of *handling* things. ... we may interpret Marx as meaning that ... the pursuit of knowledge is not, as has been thought, one in which the object is constant while all the adaptation is on the part of the knower. On the contrary, both subject and object, both the knower and the thing known, are in a continual process of mutual adaptation. He calls the process 'dialectical' because it is never fully completed.

It was natural for Marx to turn to early original sources ranging from Newton to Lagrange; indeed, McLellan observes that Marx had an 'absorbing passion for history' [8, p. 2]. Unfortunately, Marx seemed to be unaware of the enormous development of calculus during his own century — the latest original source he examines is the 1813 version of Lagrange's *Théorie des fonctions analytiques*. Beyond Lagrange, Marx relied on a fairly standard textbook by Boucharlat [1]. Marx's ignorance of this crucial period is in part due to his living in England, where, for a long time, mathematical developments in Europe had been ignored as a result of the dispute between the camps of Newton and Leibnitz. Thus the English texts of the time tended to follow an outdated exposition of mathematics. Moreover, Marx's mathematical mentor, the English translator of the *Communist Manifesto*, Samuel Moore, was also unaware of contemporary developments.

In his three history manuscripts, Marx attempted to delineate the history of calculus by examining the special case of the history of the derivative of a product. One of these manuscripts is concerned mainly with basic series expansions (Taylor and MacLaurin), after which Marx changes tack to consider Lagrange. The last historical manuscript is essentially an appendix to the first with a focus on a perceived ambiguity in the terms 'limit' and 'limit value'.

Throughout these writings, Marx relied on a detailed examination of a couple of special cases, extracting generalities from specifics, to guide his understanding.

On the concept of the derived function

The introductory manuscript of [8], 'On the concept of the derived function', was written in 1881, just two years before Marx died. Here he set out the mechanics of calculating the derivative of a function from first principles. In particular, he showed how to find derivatives of the functions $y = ax$, a cubic function, and functions of the form $y = ax^m$, $y = a^x$, and $y = \sqrt{a^2 + x^2}$ (where 'a' denotes a constant).

Marx avoided geometry in his explanations. There is no mention of gradients of tangents to curves; everything is treated in an algebraic fashion. This is in spite

Statue of Marx and Engels in Petrozavodsk, Russia. It was unveiled on May 10, 1960 and dedicated to 40th anniversary of Karelian Labour Commune. Figures are made of bronze, and the base is made of crimson quartzite, a rare Karelian stone. The size of figures is 1.7m × 1.5m × 1.3m and the base is 1.8m × 2.2m × 1.9m. Sculptors: Efim Belostotski (main artist), Elius Fridman, Peter Osipenko. Photo taken by C.T. Lenard in 2001.

of the fact that one of his sources [1] (quoted in [8, p. 24]) has a strong geometric flavour. Evidently Marx preferred algebraic arguments to geometric ones. He demonstrated his firm grasp of algebra with his calculation of the derivative of $y = a^x$ which is based on a loose argument that employs series expansions.

While it is clear that Marx could calculate the derivative of a function by algebraic manipulations, he was disturbed by the meaning of it all. For example, he calculated the derivative of $y = ax$ as follows: let $x_1 > x$, then

$$\begin{aligned} y_1 - y &= a(x_1 - x) \\ \frac{y_1 - y}{x_1 - x} &= a. \end{aligned}$$

Now let x_1 tend to x. When $x_1 = x$, we have $y_1 = y$ and hence $0/0 = a$. Then he wrote [8, p. 5]:

> Since in the expression $0/0$ every trace of its meaning has disappeared, we replace it with dy/dx … Thus, $dy/dx = a$.
>
> The closely held belief of some rationalising mathematicians that dy and dx are quantitatively actually only infinitely small, only approaching $0/0$, is a chimera …

We might be alarmed to find a student writing $0/0$. However, Marx thought deeply about the fundamentals of differential calculus and was well aware of what he was doing when he wrote '$0/0$'.

On the differential

The second manuscript is entitled ‘On the differential’. This is the most interesting of the papers for anyone who teaches calculus because here Marx explored the meaning of dy/dx in some depth. There are three earlier drafts of this paper in the collection; these demonstrate that Marx wrote the notes with considerable care.

In calculating the derivative of a function from first principles, Marx did not like the notion of a limit. When considering $(f(x) - f(a))/(x - a)$ he wanted to put $x = a$, after suitable cancelling or some other algebraic simplification, and write the result as $0/0$. He did not see $0/0$ as a fraction; he saw it as one symbol. In this respect, our use of dy/dx is similar. His only reservation about $0/0$ was that it does not refer to the function being differentiated (y) nor to its argument (x); hence he was prepared to allow the use of dy/dx. To emphasise the point, he often wrote ‘$0/0$ or dy/dx’ as if he were writing dy/dx reluctantly.

Another feature of Marx’s understanding of dy/dx is best illustrated by an example: suppose $y = x^m$ and $dy/dx = mx^{m-1}$. Marx referred to dy/dx as the ‘symbolic differential coefficient’ and mx^{m-1} as the ‘real differential coefficient’ [8, p. 16]. The symbolic differential coefficient is the symbol used to represent the derivative whereas the real differential coefficient is the specific function which is the derivative of the given function. Marx made heavy weather of this distinction which is discussed time and time again in [8].

A modern analogy can help us to appreciate the point that Marx is making. In computer science, we write $X := 2$ to indicate that X is a variable and 2 is the current value of this variable. This is quite different from the mathematical statement $X = 2$. Similarly, in Marx’s view, the nature of the object on the left-hand side of the equation $dy/dx = mx^{m-1}$ is different from the nature of the object on the right-hand side. The left-hand side represents an operation to be performed and the right-hand side represents the results of performing this operation on a particular function.

Having arrived at this conclusion, he then discussed, at length, the meaning of $d(uz)/dx = zdu/dx + udz/dx$. For Marx, the very nature of this equation needs explanation. We cannot say that the right-hand side is the result of the differentiation on the left-hand side because we are not told how either u or z depend on x. He described this equation as a ‘symbolic operational equation’ [8, p. 26]. In modern terminology, he is saying that this is an operator equation and he is concerned about the meaning of two operators being equal. Although Marx tried to teach himself the basics of calculus on his own, the depth and sophistication of his thinking is impressive.

We know that Marx had access to some edition of the work by Bourcharlat [1] in French or English because he quoted from this work. In preparing this paper, we had access only to the French edition listed in the bibliography below. Bourchalat made it very clear, in the opening pages of the work, that one uses limits to calculate the derivative of a function from first principles [1, p. 2]. It appears that Marx deliberately rejected these arguments.

On the history of differential calculus

Marx did not state it explicitly, but his historical manuscripts can be read as following a thesis-antithesis-synthesis model as he moved from Newton to Leibnitz to D'Alembert to Lagrange. His own treatment of the differential can then be seen as a natural progression from these eighteenth century authors even though his history manuscripts post-date his writings on the differential. Glivenko recognised Marx's approach and noted that Marx solved this question by examining the transition from algebra to differential calculus in mathematics dialectically [3].

The first of the history manuscripts begins by comparing the differential of a product as treated by Newton and Leibnitz. Despite their different approaches, Marx concluded that both Newton and Leibnitz 'valued differential expressions from the beginning as operational formulae' [8, p. 78]. Detailed examples treated algebraically following the manner of Newton and Leibnitz then provided Marx with his empirical evidence. Turning to a general treatment, Marx stated [8, p. 84]:

> The general direction of the algebraic method which we have applied may be expressed as follows: Given $f(x)$, first develop the 'preliminary derivative', which we would like to call $f^1(x)$: . . . .

In the second history manuscript Marx identified three phases: 'Mystical Differential Calculus', 'Rational Differential Calculus' in which D'Alembert is seen to make the 'fundamental correction' to Newton and Leibnitz, and the 'Purely Algebraic Differential Calculus' of Lagrange. By now Marx has reached what he believed to be the correct view [8, p. 99]:

> Lagrange takes as his immediate starting point for the algebraicisation of the differential calculus the theorem of *Taylor outlived by Newton and the Newtonians* which in fact is the most general, comprehensive theorem and at the same time operational formula of differential calculus, namely the series expansion, expressed in symbolic differential coefficients, of y_1 or $f(x+h)$ . . .

The titles given to these phases reveal how Marx viewed the historical development of calculus.

Two other manuscripts, one unfinished, deal with the theorems of Taylor and Maclaurin, and a further discussion of Lagrange's theory of derived functions.

The lacuna in Marx's knowledge means that he missed the 'antithesis-synthesis' which occurred after Lagrange, that is, the rigorisation of the logical foundation of calculus. As a result, Marx embarked on his own dialectical program in reaction to the deficiencies he perceived in the original works of Newton and others, and, in isolation, he reached conclusions different from those commonly accepted. Nevertheless, Marx made a valiant attempt in isolation, and we can only guess at how his study might have ended had he been fully aware of contemporary texts.

Why was Marx interested in calculus?

According to McLellan, Marx's performance in mathematics at school was 'weak' [11, p. 8]; however, Kennedy states that "Marx's Gymnasium certificate said that he had 'a good knowledge of mathematics'" [4, p. 305]. In any case, Marx did not pursue mathematics at university. There must have been a strong motivation for turning his mind, so late in life, to a technical subject such as calculus in which he had no prior training.

The manuscripts [8] do not give the reader any idea as to why he was studying calculus. To find Marx's motivations, we must look elsewhere.

Struik [16] argues that Marx studied calculus for purely intellectual reasons. 'Marx worked on mathematics in spare hours, for relaxation, often in hours of sickness ... Struck by the unsatisfactory formulations in [the books available to him] he tried in [a] characteristic way to straighten out the difficulties for himself' [16, p. 187]. Kennedy agrees that Marx was 'interested in mathematics for its own sake' [4, p. 306]. This view is supported by Marx himself to some extent; in a letter to Engels on 6 July 1863, Marx wrote: 'My spare time is now devoted to differential and integral calculus'. However, in the same letter, Marx suggested that calculus would be 'almost essential' to Engels' military studies. Although Marx's notes deal with theoretical aspects of calculus, at least he had an eye on potential applications [9].

Leon Smolinski [15] examines the mathematical manuscripts of Karl Marx to test the 'widely held view' that it was Marx's influence that delayed for decades the 'development of mathematical economics in the economic systems of the Soviet type' [15, p. 1189]. This, in turn, was said to have adversely affected the efficiency with which they operated. Smolinski found no evidence for the two opposing views that: (1) cast Marx as a mathematical illiterate; and (2) claimed that he made creative contributions to mathematics.

Smolinski argues that Marx's studies in mathematics began in 1858 with a study of algebra, and he began to study calculus systematically in 1863. He returned to the study of calculus in 1870 to keep abreast with the mathematical school of political economy. Yet despite his original intent, Marx made little application of mathematics to economics and he quickly became interested in mathematics 'primarily for its own sake' [15, p. 1193]. In his own economic writings, Marx turned to arithmetical examples rather than formal mathematical explanations. Smolinski can find no evidence in Marx's writings that he was opposed to mathematical economics. Indeed he made pioneering proposals for mathematical analysis of the business cycle and considered that a science could only become developed when it made use of mathematics. In the early 1920s Soviet model builders took Marx's lead and turned to mathematics, most notably that of Groman Kondratief. Almost without exception, these mathematical economists were critical of Joseph Stalin's forced industrialisation of the early 1930s, and incurred the wrath of Stalin. Some, like Kondratief, paid the ultimate penalty, others received long terms of imprisonment, and others abandoned economics altogether.

We offer another explanation for Marx's motivation. On 11 January 1858, he wrote to Engels [9]:

> In elaborating the principles of economics I have been so damnably held up by errors in calculation that in despair I have applied myself to a rapid revision of algebra. I have never felt at home with arithmetic. But by making a detour via algebra, I shall quickly get back into the way of things

Perhaps in pursuing his economic studies Marx revised his algebra, and from there it was a natural step to explore the basics of differential calculus.

The manuscripts offer no evidence that Marx intended to publish his notes. That he revised his work so extensively is typical of his approach to learning in any subject. During the last decade of his life, 'Marx filled in his microscopic handwriting almost three thousand pages — these manuscripts comprising almost exclusively notes on his reading. In his later years this reading became obsessional: he no longer had the power to create, but at least he could absorb' [11, p. 396].

Commentaries on Marx's mathematics

Several prominent authors found Marx's mathematics sophisticated enough to warrant attention. Glivenko [3] argues that Marx's approach to the concept of the derivative is similar to the approach taken by Hadamard in his *Cours d'Analyse* about 50 years later, and the same point is made in Kennedy [5]. Struik [16] makes a strong case, discounting the hagiography, for Marx as an original contributor to the reconciliation of the different opinions over the meaning of such a basic notion as differential. Kennedy [4] discusses how Marx interprets differentiation as a dialectical process.

There are three commentaries (Kol'man [6], Kol'man and Yanovskaya [7], and Smith [14]) bound in [8]. These authors argue for a dialectical approach to mathematics and science in general. For interesting background information on Kol'man, see Seneta [13].

In an extensive paper [10], Matthews explores the link between Marx's mathematical manuscripts and mathematical economics.

Conclusions

It is not well known, by mathematicians, economists, or historians, that Marx was interested in calculus; McLellan's now standard biography of Marx [11] makes no mention of the manuscripts [8]. The aim of this article is to inform readers of the *Gazette* about Marx's efforts to master differential calculus.

When writing his notes, Marx was teaching himself about differential calculus. He demonstrated a clear understanding of the mechanics of calculating the derivative of a function from first principles. He preferred algebraic arguments to geometric arguments. He had a clear understanding of the chain rule [8, p. 60], he dealt with Leibniz's rule for differentiating a product of several functions in a few lines

[8, p. 22], he thought deeply about the meaning of the derivative, and he was very particular about the meaning of mathematical symbols used in differential calculus.

Marx wrestled with the issues surrounding 0/0 as did many mathematicians before him [2]. He did not approve of limiting arguments. Perhaps he regarded the limiting arguments as set forth in standard texts such like [1] as insufficiently rigorous. He would have the same objection to many calculus texts these days too. Marx might prove to be a difficult student in a contemporary calculus lecture group — but tutorial discussions would not be boring.

Marx would be disappointed that, more than 120 years later, to write $dy/dx = 0/0$ still does not enter our skulls.

Acknowledgements

We are very grateful to Ms Anna Regerand, Chief Executive, External Relations Department, Petrozavodsk City Administration, Petrozavodsk, Russia for the information about the statue of Marx and Engels. We thank the referees for their helpful comments.

References

[1] Boucharlat, J.-L. (1820). *Élémens de Calcul Différential et de Calcul Intégral*, Seconde Éd., Paris: Mme. V. Courcier.
[2] Edwards, C.H. (1979). *The Historical Development of the Calculus.* Springer-Verlag, New York.
[3] Glivenko, V. (1935). Der Differentialbegriff bei Marx und Hadamard. *Unter dem Banner des Marxismus*, 102–110.
[4] Kennedy, H. (1977). Karl Marx and the foundations of differential calculus. *Historia Mathematica*, **4,** 303–318.
[5] Kennedy, H. (1982). Marx, Peano, and differentials. *Science & Nature*, **5,** 39–42.
[6] Kol'man, E. (1983). Karl Marx and Mathematics: on the 'Mathematical Manuscripts' of Marx. in [8], 256–270.
[7] Kol'man, E. and Yanovskaya, S.A. (1983). Hegel and mathematics. in [8], 256–270.
[8] Marx, K. (1983). *Mathematical Manuscripts of Karl Marx.* New Park Publications, London. Available at http://www.marxists.org/archive/marx/works/1881/mathematical-manuscripts (accessed 30 July 2009).
[9] Marx, K. *Collected Correspondence.* Available at http://www.marxists.org/archive/marx/letters/ (accessed 30 July 2009).
[10] Matthews, P.H. (2002). The dialectics of differentiation: Marx's mathematical manuscripts and their relation to his economics. Middlebury College Working Paper Series, Number 0203, Middlebury College, Department of Economics. Avaliable at http://ideas.repec.org/p/mdl/mdlpap/0203.html (accessed 30 July 2009).
[11] McLellan, D. (2006) *Karl Marx: A Biography*, 3rd edn. Palgrave Macmillan. Hampshire.
[12] Russell, B. (1961, repr. 1995). *History of Western Philosophy*, 2nd edn. Routledge, London.
[13] Seneta, E. (2004). Mathematics, religion, and Marxism in the Soviet Union in the 1930s. *Historia Mathematica*, **31,** 337–364.
[14] Smith, C. (1983). Hegel, Marx and the calculus. in [8], 256–270.
[15] Smolinski, L. (1973). Karl Marx and mathematical economics. *Journal of Political Economy*, **81,** 1189–1204.
[16] Struik, D.J. (1948). Marx and mathematics. *Science and Society*, **12,** 181–196.