A correspondence note on Myerson's 'Irrationality via well-ordering'

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I thoroughly enjoyed Myerson's article [2] on methods of proving irrationality via the well-ordering principle. In this note, I point out a second method by which the well-ordering principle may be used to prove Myerson's Theorem 2. This approach is a generalisation of MacHale's recent proof [1] that $\sqrt{2}$ is irrational.

Theorem 1. *For $m \in \mathbb{N}$, $\sqrt{m}$ is irrational if it is not an integer.*

Proof. First, we prove the result for squarefree m. We consider the set

$$\{a + b \mid a, b \in \mathbb{N}, a^2 = mb^2\}.$$

By the well-ordering principle for $\mathbb{N}$, this set has a minimal element $a_0 + b_0$. If m is squarefree, the condition $a_0^2 = mb_0^2$ guarantees that m divides a_0. Thus, $a_0 = m\ell$ for some $\ell \in \mathbb{N}$. But then, $m^2\ell^2 = a_0^2 = mb_0^2$, from which it follows that m divides b_0. Writing $b_0 = mr$, we have $m^3r^2 = m^2\ell^2$. It follows that $mr^2 = \ell^2$. As $a_0 + b_0 = m(\ell + r)$, this is a contradiction to the minimality of $a_0 + b_0$.

Now, if m is not squarefree, we may write $m = m_1^2 m_0$ for m_0 squarefree. We have $\sqrt{m_0}$ irrational by the earlier argument, so $\sqrt{m} = \sqrt{m_1^2 m_0} = m_1\sqrt{m_0}$ is irrational, as well.

References

[1] MacHale, D. (2008). The well-ordering principle for $\mathbb{N}$. *Math. Gaz.* **92,** 257–259.
[2] Myerson, G. (2008). Irrationality via well-ordering. *Gaz. Aust. Math. Soc.* **35,** 121–125.

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