Communications

The essential elements of mathematics: a personal reflection

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The essence of Mathematics is three-fold. It encompasses truth, beauty and power. Mathematics attracts us because of its beauty, because with it we can confidently speak the truth, because it is 'unreasonably effective' in describing the world.

The description by any individual of the essential elements of Mathematics is unavoidably influenced by that individual's experience of Mathematics. This document is written from my personal viewpoint, and so reflects my own experiences.

It was a mixture of things that *attracted* me to Mathematics, including especially

- the way Mathematics helped explain how things worked in the world;
- the excitement of solving previously unsolved problems;
- the sheer beauty of mathematical patterns and structure;

and this same mix *sustained* my commitment to the discipline.

The power of Mathematics

If I were forced to choose the single most important aspect of Mathematics, that epitomizes it, then I would choose its power. If I were asked to nominate the most important outcome for a student of Mathematics, at any level, I would say: *an understanding of the power of Mathematics, or differently put, an understanding of the 'unreasonable effectiveness' of Mathematics in making sense of the world.*

Many mathematicians from all ages have professed this same conviction. The first record of this seems to date back to Pythagoras around 500 BC. Pythagoras believed that all parts of the world were governed by mathematical principles, and is credited as saying 'Mathematics is the way to understand the universe' [1]. Some two thousand years later Galileo, who discovered that the earth rotates around the sun, held the view that 'the laws of Nature are written in the language of mathematics' [2, p. 171].

In 1960 the physicist and Nobel Laureate, Eugene Wigner described Mathematics as being 'unreasonably effective' [3] because of its sometimes surprising power in solving 'real-world problems'. Wigner argued that 'the ability of mathematics to

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successfully predict events in [science] cannot be a coincidence, but must reflect some larger or deeper or simpler truth in both' [1].

The dual aims of mathematics education in schools must be

- to provide students with the personal confidence and competence to solve mathematical problems by developing in them a mathematical way of thinking, and also
- to give students an awareness of the power and influence of Mathematics in their everyday life.

The crucial role of Mathematics in our technological society is recognised far beyond the ranks of the practitioners (academics, teachers, and those using Mathematics in industry or commerce). For example, E.E. David, the former President of Exxon Research and Engineering wrote: 'too few people recognise that the high technology that is so celebrated today is essentially mathematical technology' and Barry Jones, former Australian Federal Minister for Science, wrote in 1988 when he held that position: 'Science and Maths are part of the language and the only way to avoid our becoming mere passive users of technology is to ensure that more children are educated in mathematics and science' [4].

However it is not only in the area of technology that Mathematics is important. Its universality is difficult to capture in a few words. As Solomon Marcus, an eminent Romanian mathematician, writer and educator said: 'all professions need a mathematical training related to their way of thinking and a mathematical background. Mathematics has a kind of universality and any form of education should exploit this fact' [5].

It is not clear how to translate this sentiment into a prescription for school mathematics curricula. However, I consider the most important outcome from a mathematics education to be an automatic expectation by students that mathematical thinking will play a key role in their understanding and problem-solving in every part of their lives. Solomon Marcus attempted to list various types of mathematical thinking. Each of us could make a similar list, and I am in broad agreement with Marcus' list and views: 'combinatorial thinking, recursive thinking, algorithmic thinking, step-by-step thinking, deductive thinking, inductive thinking, thinking by analogies, probabilistic thinking, are only a part of the many types of mathematical thinking. They are essential not only in Mathematics, not only in science, but also in all aspects of life, even in the absence of mathematical jargon. ... Mathematics is a part of the cultural heritage of mankind' [5]. Each of these types of thinking, when explored, is very wide-ranging, and most of them should feature in the mathematical experience and training of school students.

A school Mathematics curriculum is designed to prepare children for life in a world that may be different from anything we can imagine. The ability to think mathematically is the best gift we can give our children from their mathematics education.

For example, I studied Mathematics a long time before the modern Information Technology revolution. Indeed, my children find it difficult to believe that I never had an electronic calculator before they were born. However, by the time I was in high school much of the preparation was in place for this IT revolution. I just did not know about it. For example,

- already in 1948, Claude Shannon, now called the Father of Information Theory, had established the mathematical basis for electronic communication;
- W. Edwards Deming's statistical methods, that assisted the post-war recovery of the Japanese economy, had laid the foundations for the mathematical theory of quality management and quality control so fundamental for industry today;
- R.W. Hamming had completed much of his mathematical work on error-correcting codes that, perhaps unknowingly, we use every day for reliable use of our personal computers.

I chose to study Mathematics without knowing anything about these incredible Mathematical breakthroughs, but I *had* seen in school the power of Mathematics in describing many natural phenomena. As a rather simple example, I remember being quite incredulous of my physics teacher's claims that we could predict mathematically the exact position of our reflection in a mirror. I needed to see this with my own eyes in my first laboratory experiment before I was willing to believe it. The fact that I learned nothing at school about the breakthroughs in IT is of no consequence. What did matter was that I saw mathematical principles applied successfully to describe a range of phenomena within my everyday experience.

When we think of the crucial impact of Mathematics in today's world, one of the principal areas of impact is computer and communications technology mentioned above. However, there are many others.

- Modern weather forecasting is based on the output of large-scale computer models which, in turn, are based on fundamental mathematical principles.
- (quoting from a public lecture I gave at the 2003 Malaysian Science and Technology Congress): In the areas of health and biology, mathematical and statistical methods are the foundations for biostatistics, epidemiology and randomised clinical trials that, according to a recent report produced in the US, 'have been cornerstones of the systematic attack on human disease that has dramatically increased life expectancy in advanced societies during the past half century' [6]. It is not widely known that Florence Nightingale (1820–1910), who is best known for her pioneering of modern nursing and the reform of hospitals, and in particular for her care of British soldiers wounded in the Crimean War (1854–1856), is also regarded as a pioneer of epidemiological methods for her use of public health statistics. She was a highly respected member of the (Royal) Statistical Society, and indeed she was the first woman to be elected a member. She is quoted as saying: 'to understand God's thoughts we must study Statistics, for these are the measure of his purpose' [7, Volume II, Chapter xiii].
- Some years ago, we learned with excitement and wonder that the human genome has been sequenced successfully. We anticipate that recent progress in molecular biology and genetics will lead to *rapid advances in understanding fundamental life processes at the molecular level*. Long-term goals of this research include *the alleviation of malnutrition and starvation by improving agriculturally important plant species and domestic animals, the improvement of public health, and better defence against bio-terrorism*. Statistical and computational methods have played and will continue to play an important role [6] in this research.

- Global competition and increased customer and government requirements are transforming the world of business. A wide range of difficult mathematical challenges is associated with the problems of analysing large complex-structured data sets; using simulations to reduce requirements for physical testing of new products; modelling and interpreting data related to reliability and safety of existing products; and, in general, developing new and emerging technologies, such as wireless communication technology.

In order to help students appreciate how crucial Mathematics will be for their future, it is essential that they learn through their school experience how Mathematics has transformed society over the past century (at least) and how it underpins the society in which they now live.

Mathematics and truth

I asked a young graduate student in Teheran why she was studying Mathematics. Her answer has stayed with me over the years. She said 'because with Mathematics I know that I speak the truth'. The precision of mathematical language, and the requirement of rigorous logical reasoning, underpin the unique claim of Mathematics to absolute truth.

In mathematical discourse the demand is both for clarity in the assumptions made and also for clarity in the logical rules used in making deductions from those assumptions.

Thus, an education in mathematical discourse is an excellent training for someone who aims to become an outstanding lawyer! Other examples abound where mathematically based discourse is essential to a discipline or profession. For example, in cosmology, deductions are made about the past or future that are based on mathematical reasoning, and are conditional on the truth of equations describing current observations.

Part of the gift of a mathematics education for students is the power of critical and logical thinking. Many years ago, H.G. Wells displayed great foresight in asserting that 'statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write'[2], and in my view his assertion applies to all mathematical thinking, not just statistical thinking. Moreover, in the words of the contemporary mathematician Hyman Bass: 'the characteristic that distinguishes mathematics from all other sciences is the nature of mathematical knowledge and its certification by means of mathematical proof' [8].

It is essential that students learn critical thinking, and facilitating this learning is a key responsibility of a mathematics education. In the past, school students gained an understanding of the nature of proof from their study of Euclidean geometry. They learned by experience what it meant to prove a statement about a general class of geometrical objects. That is to say, based on certain assumptions and using certain allowable rules of deduction, students learned that they could prove a general statement that held true whenever the assumptions were satisfied. This

[2]See, for example, www.keypress.com/fathom/quotes.html (accessed 11 November 2007). This quote appears in many other places on the internet.

is a sophisticated concept for a student to grasp. An example is Pythagoras's Theorem: 'the square of the hypotenuse of a right angle triangle is equal to the sum of the squares on the other two sides'. Although Chinese and Babylonian mathematicians had used this result for almost a thousand years before Pythagoras lived, Pythagoras's contribution, and the reason why this theorem has his name, was to prove that it was true for *every* right-angled triangle.

Whether this understanding of the nature of mathematical proof is learned through studying geometry or from some other area of Mathematics is not important. What is important is that students of Mathematics have the opportunity to learn critical and logical thinking, and that they do learn it.

The beauty of Mathematics

Beauty is a major part of our fascination with mathematical patterns, and these patterns are used effectively in school mathematics courses to attract students and develop their problem-solving skills.

We also perceive as beautiful the simple ideas and concepts that inspired John Nash's mathematical work on non-competitive games that had enormous influence on economic theory, and resulted in Nash's award of a Nobel Prize.

However, fascination turns to awe when confronted with the proof by Andrew Wiles in 1993 of Fermat's Last Theorem: there are no integer solutions x, y, z to the equation $x^m + y^m = z^m$ for any $m > 2$. This was a massive and difficult proof; proving at last an assertion of Pierre de Fermat more than 300 years earlier in the margin of his copy of Diophantus's *Arithmetica*. The proof had both eluded and fascinated mathematicians throughout those years.

My experience of beauty in Mathematics is bound up with feelings of awe and admiration for the power that underlies the beauty. A wonderful quote from the mathematician and philosopher Bertrand Russell gives further expression to this: 'Mathematics, rightly viewed, possesses not only truth, but supreme beauty; a beauty cold and austere, like that of sculpture, . . . , and capable of a stern perfection such as only the greatest art can show' [9, Chapter 4].

Conclusion

In this paper I have discussed my convictions about the fundamental elements of a mathematics education. I have said almost nothing about the content of school mathematics curricula for Years 1 to 10[3]. Indeed those with experience in Mathematics teaching at school level will have greater expertise than I in designing a curriculum to achieve the outcomes deemed essential for a successful mathematics education for these years. The current discourse, for which I hope this paper will prove useful, is aimed at achieving agreement on the essentials of the school mathematics experience and its outcomes.

I have argued that the essential outcomes for students from their mathematics education must include

- an awareness of the power of Mathematics to make sense of the world;

[3] The focus of the *Framework of Essential Learning* developed by the VCAA.

- a personal confidence and competence to solve mathematical problems;
- an understanding of the nature of proof in Mathematics;
- an appreciation of the beauty of Mathematics;
- the ability to think logically and critically, learned through Mathematics; and
- an automatic expectation that mathematical thinking will play a key role in their understanding and problem-solving in every part of their lives.

I am aware that school mathematics curricula often divide Mathematics into a number of areas or strands: number, chance and data, space, etc. While this may be appropriate for teaching and assessment, it runs the risk of students developing an incorrect and fragmented view of Mathematics. A wise curriculum designer will ensure that students meet striking examples showing how apparently heterogeneous fields can interact strongly with each other. As an example, Solomon Marcus [5], in addressing this issue, observed that the mathematical object called a Mobius strip over the past few decades *became a basic point of reference in anthropology (Claude Levi-Strauss), in art (M.C. Escher), in biology (Jesper Hoffmeyer) and in many other fields.*

Mathematics has been in the vanguard of all important technological and social change. Because of this, our children, who are after all our future, need strong mathematical skills. This is essential for the future health and wealth of Australia. My vision is for our children to know that Mathematics is both beautiful and powerful, and moreover, to understand, if only in part, the 'unreasonable effectiveness' of Mathematics in describing the world.

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