On Fourier and Hankel Sampling

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Abstract

We use the Paley–Wiener theorem for the Fourier and Hankel transforms to compare Fourier and Hankel Sampling.

1 Introduction

The Paley–Wiener theorem for the Fourier transform is an indispensable tool in Shannon sampling, see e.g., [12]. Assume the Fourier transform $\mathcal{F}f$ of $f \in L^2(\mathbb{R})$ has support in $[-1,1]$, that is, f is a Fourier bandlimited function. Shannon's sampling theorem (also called the Whittaker–Kotel'nikov–Shannon sampling theorem) then yields the well-known sampling formula

$$f(z) = \sum_{n=-\infty}^{\infty} f(\pi n) \frac{\sin(z-\pi n)}{z-\pi n} \qquad (z \in \mathbb{C}), \tag{1}$$

with absolute and uniform convergence on compact subsets of $\mathbb{C}$. To prove (1), one uses the Fourier inversion formula, the Fourier series for $\mathcal{F}f$ and the formula: $\sin t/t = 1/2\int_{-1}^{1} e^{ixt}dx$.

We define the Hankel transform $h_\nu(f)$ of order ν ($\nu > -\frac{1}{2}$) of a function $f \in L^1(\mathbb{R}_+, t^{2\nu+1}dt)$ by

$$h_\nu f(x) = \int_0^\infty f(x)(xt)^{-\nu} J_\nu(xt) t^{2\nu+1} dt \qquad (x \in \mathbb{R}_+),$$

where J_ν denotes the Bessel function of the first kind

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}.$$

The Hankel transform extends to an isometric isomorphism of $L^2(\mathbb{R}_+, t^{2\nu+1}dt)$ onto itself, with symmetric inverse: $h_\nu^{-1} = h_\nu$. Other versions and definitions of Hankel transforms can be found in the literature. We note that our Hankel transform for $\nu = n/2$, with $n \in \mathbb{N}$, reduces to the classical Fourier transform for radial functions in $\mathbb{R}^n$.

Kramer's sampling theorem, see e.g., [3], gives a sampling formula for a Hankel bandlimited function, that is, a function whose Hankel transform has compact support, see also (2). An obvious question is: for which class of functions is each of the sampling formulae valid? The relationship between the two sampling formulae has been discussed in several papers, for further details and the history of this problem, we refer to [2, 3, 4, 6, 12] and references therein.

The classical approach to the problem is by manipulating integral formulae. In this short note, we use the Paley–Wiener theorems for the Fourier and Hankel transforms to give an

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easy proof of results in [3]. Surprisingly, this approach seems new. The classical Paley–Wiener theorem can be found in [8], see also [10, Theorem 19.3]. The earliest proof of a Paley–Wiener theorem for the Hankel transform [5] was published 50 years ago by the Australian mathematician J. L. Griffith, see also [11], but unfortunately this paper seems almost unknown, even to experts.

Theorem 1 (J.L. Griffith) *Let $\nu > -\frac{1}{2}$ and $1 = \frac{1}{p} + \frac{1}{q}$. Let f be an even entire function of exponential type 1. If $1 < p \leq 2$ and $t^{\nu+1/2} f(t) \in L^p(0, \infty)$, then f can be represented by*

$$f(z) = \int_0^1 (xz)^{-\nu} J_\nu(xz)\phi(x)dx \qquad (z \in \mathbb{C}),$$

with $x^{-\nu-1/2}\phi(x) \in L^q(0, 1)$. Conversely, if f has this representation and $x^{-\nu-1/2}\phi(x) \in L^p(0, 1)$, $1 < p \leq 2$, then f is an even entire function of exponential type 1 such that $t^{\nu+1/2} f(t) \in L^q(0, \infty)$.

2 Bandlimited functions

A Hankel bandlimited function is in [3] defined as a function f with integral representation

$$f(t) = \int_0^1 J_\nu(xt)H(x)xdx \qquad (t \in \mathbb{R}_+), \tag{2}$$

for a function H. Assume $\sqrt{x}H(x) \in L^2(0, 1)$, and define $\phi(x) := x^{\nu+1}H(x)$. Then $x^{-\nu-\frac{1}{2}}\phi(x) \in L^2(0, 1)$, and

$$f(t) = t^\nu \int_0^1 (xt)^{-\nu} J_\nu(xt)\phi(x)dx, \qquad (t \in \mathbb{R}_+).$$

The function $g(t) := t^{-\nu} f(t)$ is by Theorem 1 the restriction to $\mathbb{R}$ of an even entire function of exponential type 1 such that $t^{\nu+1/2} g(t) \in L^2(0, \infty)$. It follows that $g \in L^2(0, \infty)$, and by the (Fourier) Paley–Wiener theorem that

$$f(t) = t^\nu \int_{-1}^1 F(x)e^{ixt}dx, \qquad (t \in \mathbb{R}_+), \tag{3}$$

for some (even) $F \in L^2(-1, 1)$, which is (4) in [3].

Conversely, assume f can be written as in (3) for some even $F \in L^2(-1, 1)$. Then $g(t) := t^{-\nu} f(t)$ is Fourier bandlimited and is thus an even entire function of exponential type 1 by the (Fourier) Paley–Wiener theorem. Assume $t^{\nu+\frac{1}{2}} g(t) \in L^2(0, \infty)$. By Theorem 1, f can be written as

$$f(t) = t^\nu \int_0^1 (xt)^{-\nu} J_\nu(xt)\phi(x)dx \qquad (t \in \mathbb{R}_+),$$

with $x^{-\nu-1/2}\phi(x) \in L^2(0, 1)$. We get (2) with $\sqrt{x}H(x) \in L^2(0, 1)$ if we define $H(x) := x^{-\nu-1}\phi(x)$, and f is thus Hankel bandlimited.

In particular, let $\nu = n+\frac{1}{2}$ with $n \in \mathbb{N}\cup\{0\}$, and assume that F is $n+1$ times differentiable with $F^{(k)}$, $k = 0, \ldots, n$, continuous, $F^{(k)}(1) = 0$, $k = 0, \ldots, n$, and $F^{(n+1)} \in L^2(-1, 1)$. Then

$$t^{\nu+\frac{1}{2}} g(t) = t^{n+1} g(t) = (-i)^{n+1} \int_{-1}^1 F^{(n+1)}(x)e^{ixt}dx \qquad (t \in \mathbb{R}_+),$$

and thus $t^{\nu+\frac{1}{2}} g(t) \in L^2(0, \infty)$. This essentially gives [3, Theorem 4] with the extra information that we know our $\sqrt{x}H(x) \in L^2(0, 1)$.

3 The Dunkl transform

Finally a few words about the Dunkl transform. The Dunkl transform on the real line reduces to our Hankel transform when restricted to even functions, see [9] for details. A new elementary proof of Theorem 1 can then be deduced from [1].

Many (new) proofs of Paley–Wiener theorems for the general Dunkl transform have appeared recently, see [1] and the references therein. We note in particular that the proof in [7] is by reduction to the one-dimensional even case, and thus more or less follows from the work of J. L. Griffith.

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