Mathellaneous

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The Joy of SET

1 What is SET?

In his monumental compendium, *The World of Mathematics* [3], James Newman states that

> *Games are among the most interesting creations of the human mind, and the analysis of their structure is full of adventure and surprises.*

Certainly many a mathematician, from amateurs to professionals of some repute, have dabbled in the analysis of games. A particular one which has captured my own mathematical curiosity of recent times is the delightful, though little known, card game called SET[1]. Some of the *Gazette's* readership may already be well-acquainted with the game, but for the unenlightened, let us begin to answer the question, "What exactly is SET?"

SET[2] is a game played with a special deck of cards, each of which depicts either one, two or three objects. These objects can be any of three different shapes — oval, squiggle or diamond — and they are portrayed in one of three different colours — red, green or purple. Furthermore, the objects come in three different shadings — empty, striped or solid. One will never find two different shapes, colours or shadings on the same card. Therefore, each card can be described by the four attributes number, shape, colour and shading; and each of these attributes can take on one of three values as listed in the table below. Of course, you have probably already guessed that there are $3^4 = 81$ cards in the standard SET deck, exactly one for every possible combination of number, shape, colour and shading.

Number	Shape	Colour	Shading
1	oval	red	empty
2	squiggle	green	striped
3	diamond	purple	solid

The main aim of the game is to identify, among a number of cards dealt face up on the table, three cards which form a SET. What exactly this means is described by...

> The SET Rule: Three distinct cards form a SET if and only if they are all different or all the same with respect to number, shape, colour and shading.

For example, the three cards below form a SET because they are all the same with respect to number, all different with respect to shape, all the same with respect to colour[3], and all different with respect to shading.

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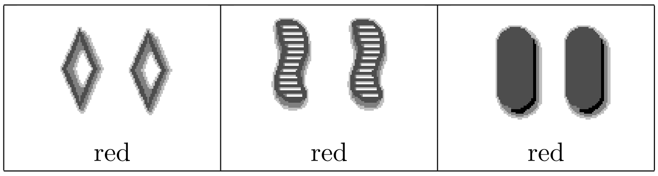
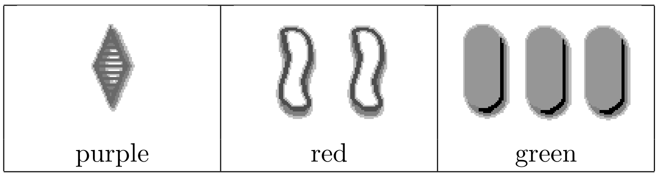
[1] SET® is a registered trademark of SET Enterprises Inc., and SET game play is protected intellectual property. The images of SET cards have been taken from The Set Game Company Homepage [4] and are used here with permission.

[2] I will continue to refer to the game SET in capital letters and reserve the lower case form to denote the well-known mathematical concept of the same name.

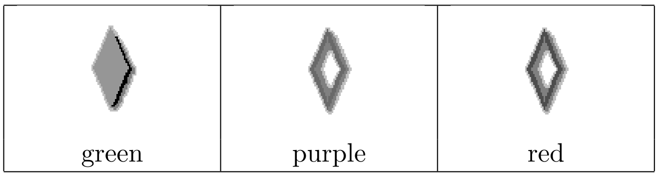
[3] Given the monochromatic nature of the *Gazette*, I have written the colours of the cards immediately below the symbols. Of course, the original SET cards come in vibrant red, green and purple and without the accompanying text.



Another example of a SET is given by the following three cards, since they are all different with respect to each of number, shape, colour and shading. Such SETs as these, in which the three cards are all different with respect to all four attributes, are affectionately known as *beasts* in some SET-playing circles.



However, the three cards shown below *do not* form a SET because they are neither all different nor all the same with respect to shading.

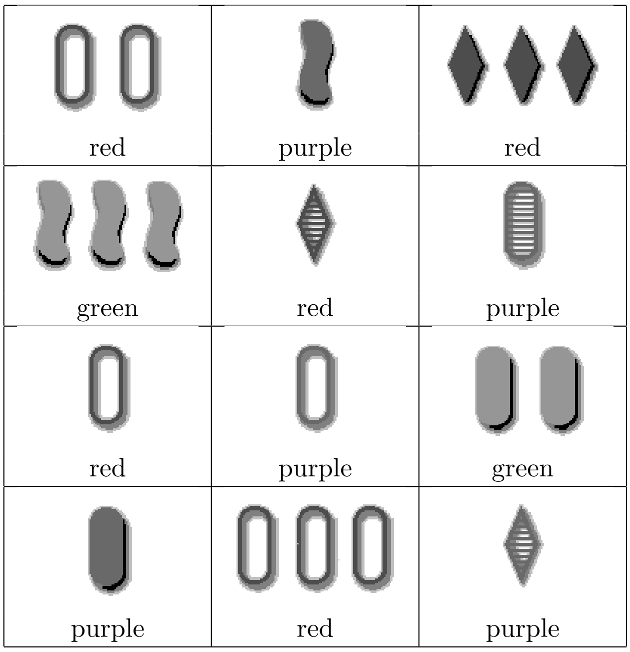


Generally, SET is a non-competitive game for one or more players and play usually proceeds as follows.

- Twelve cards are dealt face up on the table so that they can be seen by all players.
- Players must look for a SET and once found, may remove the three cards involved. Three more are then dealt in their place to restore the number of cards to twelve.
- If no SET appears, then three more cards are dealt face up, until one does appear. When this SET is removed, no more cards need be dealt, since there will be at least twelve cards remaining on the table.
- If there are no more cards to be dealt, then the game ends once no SET appears in the remaining cards on the table.

The beauty of SET lies in its simplicity. With only one fundamental rule, children as young as the age of five can enjoy the game. On the other hand, hardcore SET junkies like to record their times and I have seen people flash through the deck in under ninety seconds. From a mathematical viewpoint, the game of SET provokes a myriad of interesting problems. This article will take us on a whirlwind tour of the mathematics of SET, from finite geometry to the Fourier transform and beyond.

Problem: Find all six SETs which appear in the twelve cards pictured below.



2 Naive SET Theory

It was not long after learning the game that I heard rumours from fellow SET addicts of strange and beautiful mathematics lurking behind the cards — thus, I decided to embark on my own naive exploration of SET theory. The primary connection between SET and mathematics is the correspondence between the SET deck and $\mathbb{Z}_3^4$, the set of 4-tuples of integers modulo 3. There are many natural bijections between these two sets, although for the sake of concreteness, let us fix the correspondence to be the one shown in the table below.

$\mathbb{Z}_3$	Number	Shape	Colour	Shading
0	1	oval	red	empty
1	2	squiggle	green	striped
2	3	diamond	purple	solid

So, for example, the SET card with two ovals, coloured red and shaded solid, may be represented as the 4-tuple $(1, 0, 0, 2) \in \mathbb{Z}_3^4$. However, this is more than just a nifty shorthand for communicating SET cards, since the additive group structure of $\mathbb{Z}_3^4$ actually encapsulates the SET rule. To see this, consider the solutions to the equation $a + b + c = 0$, where a, b, c are elements of $\mathbb{Z}_3$. It should be reasonably clear that there are nine solutions for (a, b, c) given by the six permutations of $(0, 1, 2)$ as well as $(0, 0, 0)$, $(1, 1, 1)$ and $(2, 2, 2)$. In other words, three elements of $\mathbb{Z}_3$ sum to zero if and only if they are all different or all the same. Therefore, a SET consists of three cards whose representatives in $\mathbb{Z}_3^4$ sum to zero in all four components.

> The Algebraic SET Rule: Three distinct elements A, B, C of $\mathbb{Z}_3^4$ form a SET if and only if $A + B + C = 0$.

In light of the fact that we are working modulo 3, the equation $A + B + C = 0$ can be written in the equivalent form $A - B = B - C$. Thus, when three elements of $\mathbb{Z}_3^4$ form a

SET, then they actually lie on a line. Conversely, any three distinct collinear points of $\mathbb{Z}_3^4$ will satisfy the equation $A - B = B - C$, and hence, will form a SET.

> The Geometric SET Rule: Three distinct elements A, B, C of $\mathbb{Z}_3^4$ form a SET if and only if they are collinear.

These two mathematical formulations of the game provide us with familiar foundations for answering problems that arise from playing SET. For example, experienced players will know that the game usually concludes with 12, 9, or sometimes 6 cards remaining on the table with no SET among them. However, on one occasion, I was fortunate enough to spot a SET in the final six cards and, to my surprise, found that the final three cards also formed a SET. Not only did this leave the table satisfyingly clean, but it also seemed suspiciously fortuitous — I decided to explore and arrived at the following...

Theorem 1 *A game of SET cannot end with three cards.*

Proof. First note that the 81 elements of $\mathbb{Z}_3^4$ sum to 0, since they can be partitioned into 27 SETs of the form $(0, a, b, c), (1, a, b, c), (2, a, b, c)$, each of which sum to 0. Thus, no matter which 26 SETs have been removed from the deck, the final 3 cards must have a zero sum. So by the algebraic SET rule, the final three cards will also form a SET. □

On another SET-playing occasion, a friend of mine cryptically exclaimed that he had proved that a SET lay among the twelve cards, but could not see exactly where it was. His delightfully non-constructive and purely existential statement intrigued the mathematician within me so I decided to inquire. It turned out that he had hit upon the fact that 5 cards which share 2 common attributes must always contain a SET. An equivalent statement is that in a 2-dimensional version of SET, where there are only two varying attributes and the cards correspond to elements of $\mathbb{Z}_3^2$, the maximum number of cards which do not contain a SET is 4. This result was quickly confirmed given a few minutes, a pencil and the back of an envelope. However, it also prompted me to think about d-dimensional SET, where there are d varying attributes and the cards correspond to elements of $\mathbb{Z}_3^d$. And for various values of d, I wondered whether it was possible to answer the following question.

> In d-dimensional SET, what is the largest number of cards which do not contain a SET?

Combinatorialists interested in finite geometry will recognise the problem as determining the maximum size of a cap in $\mathbb{Z}_3^d$. In general, a cap is a set of points with no 3 collinear and we will define a d-cap to be a cap in $\mathbb{Z}_3^d$. The search for maximal caps is an active area of mathematical research with many interesting combinatorial techniques and results. We will learn more about maximal caps in the next section, but in the meantime, I encourage the reader to try the following problem.

Problem: Prove that a maximal 2-cap contains 4 points.

3 The Search for Maximal Caps

In this section, we will concentrate on determining the size of maximal d-caps and it will pay to think of $\mathbb{Z}_3^d$ geometrically. For example, the points of $\mathbb{Z}_3^2$ can be represented as the squares on a tic-tac-toe board, where lines are allowed to "wrap around" to the other side, à la Pac Man. So apart from the usual 8 winning tic-tac-toe lines, there are 4 extra lines which wrap around the board. However, since the enthusiastic reader will already have solved the maximal cap problem in 2-dimensions, let us turn our attention to the 3-dimensional case.

In a similar fashion, the points of $\mathbb{Z}_3^3$ can be represented as the unit cubes in a $3 \times 3 \times 3$ grid, and again, lines are allowed to "wrap around" to the other side. Note also that the 27 points can be decomposed into 3 layers, each one being a plane containing 9 points. There are many ways to do this, one for each class of parallel planes, and three such planes which partition the points of $\mathbb{Z}_3^3$ will subsequently be referred to as a *triple plane decomposition.* The determination of the size of a maximal 3-cap involves a clever application of the age old trick known as double counting. The cleverness lies in finding objects which can be double counted to give important information, and these objects turn out to be 2-pointed planes. Given a subset C of $\mathbb{Z}_3^4$, define a *2-pointed plane* to be a pair $(H, \{x, y\})$ consisting of a plane H in $\mathbb{Z}_3^3$ together with a set of two points $\{x, y\} \subseteq C \cap H$. With these definitions under our metaphorical belts, we are ready to see the proof of the following...

Theorem 2 *A maximal 3-cap contains 9 points.*

Proof. Suppose that there exists a 3-cap containing 10 points. In other words, assume that there exists a subset C of $\mathbb{Z}_3^3$ such that $|C| = 10$ and C contains no three collinear points. Note that the restriction of a 3-cap to a plane yields a 2-cap, so that every plane intersects C in at most 4 points. Therefore, given a triple plane decomposition $\mathbb{Z}_3^3 = H_1 \cup H_2 \cup H_3$, the triple $(|H_1 \cap C|, |H_2 \cap C|, |H_3 \cap C|)$ must be $(4, 4, 2)$ or $(4, 3, 3)$, up to permutation. We will refer to this triple as the *type* of a triple plane decomposition.

Now let a denote the number of triple plane decompositions of type $(4, 4, 2)$ and let b denote the number of triple plane decompositions of type $(4, 3, 3)$. Note that each line through the origin in $\mathbb{Z}_3^3$ uniquely determines the triple plane decomposition perpendicular to it. Therefore, the number of triple plane decompositions is simply equal to the number of lines through the origin. However, each line contains exactly two non-zero points, so the number of lines is simply half the number of non-zero points, namely $\frac{3^3-3}{2} = 13$. Therefore, we have the simple equation

$$a + b = 13.$$

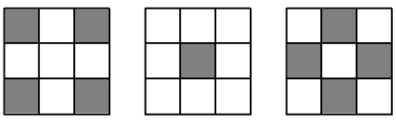
In order to derive another equation satisfied by a and b, let us double count 2-pointed planes.

- Since there are 10 points in C, there are $\binom{10}{2} = 45$ pairs of points. And it is easy to verify that through each pair of points in $\mathbb{Z}_3^3$ passes exactly 4 planes. Thus, the number of 2-pointed planes is simply $45 \times 4 = 180$.
- On the other hand, for each plane containing k points of C, there will be a contribution of $\binom{k}{2}$ to the count of 2-pointed planes. Thus, for each triple plane decomposition of type $(4, 4, 2)$, the contribution is $\binom{4}{2} + \binom{4}{2} + \binom{2}{2} = 13$, while for those of type $(4, 3, 3)$, the contribution is $\binom{4}{2} + \binom{3}{2} + \binom{3}{2} = 12$. Therefore, the number of 2-pointed planes can also be expressed as $13a + 12b$.

Of course, combining these two facts yields the equation

$$13a + 12b = 180.$$

Simultaneously solving these two equations yields the unique solution: $a = 24, b = -11$. Of course, the negativity of b is an absurdity, so we have reached the desired contradiction from which we conclude that no 3-cap exists which contains 10 points. It suffices now to verify that the following diagram represents a 3-cap with 9 points.



□

Although the proof given here is not the simplest, it has the advantage of generalising to show that a maximal 4-cap contains 20 points. The proof of this statement, however, will involve *triple hyperplane decompositions* rather than triple plane decompositions, *2-pointed hyperplanes* rather than 2-pointed planes, as well as *3-pointed hyperplanes*, all of which are defined analogously.

Theorem 3 *A maximal 4-cap contains 20 points.*

Proof. Once again, let us begin by supposing that there exists a 4-cap containing 21 points. In other words, assume that there exists a subset C of $\mathbb{Z}_3^4$ such that $|C| = 21$ and C contains no three collinear points. Since the restriction of a 4-cap to a hyperplane yields a 3-cap, every hyperplane intersects C in at most 9 points. In particular, every triple hyperplane decomposition must have one of the following types:

$$(9,9,3),(9,8,4),(9,7,5),(9,6,6),(8,8,5),(8,7,6),(7,7,7).$$

Let x_{ijk} denote the number of triple hyperplane decompositions of type (i,j,k). Using the argument above, there are 40 triple hyperplane decompositions of $\mathbb{Z}_3^4$, so we have the equation

$$x_{993} + x_{984} + x_{975} + x_{966} + x_{885} + x_{876} + x_{777} = 40.$$

A similar double count of the number of 2-pointed hyperplanes provides us with the equation

$$75x_{993} + 70x_{984} + 67x_{975} + 66x_{966} + 66x_{885} + 64x_{876} + 63x_{777} = 2730.$$

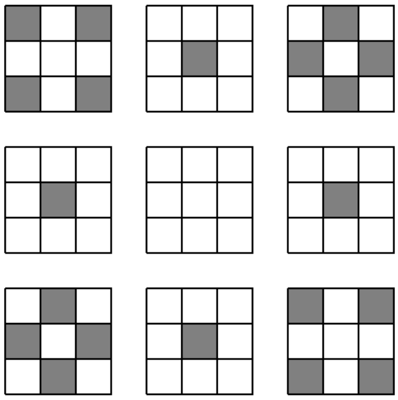
And an entirely analogous double count of the number of 3-pointed hyperplanes gives us still another equation

$$169x_{993} + 144x_{984} + 129x_{975} + 124x_{966} + 122x_{885} + 111x_{876} + 105x_{777} = 5320.$$

Alas, we still only have 3 equations in 7 variables, a far cry from the number we require to actually solve the system. Fortunately, our variables must all be non-negative integers, a restriction which we can take full advantage of. Taking a clever linear combination of the three equations which we have derived gives the following.

$$\begin{array}{ccccccccccccc} & & 5x_{984} & + & 8x_{975} & + & 9x_{966} & + & 3x_{885} & + & 2x_{876} & = & 0 \\ 12x_{993} & + & 7x_{984} & + & 4x_{975} & + & 3x_{966} & + & 3x_{885} & + & x_{876} & = & 210. \end{array}$$

Due to the non-negativity of the variables, the first equation implies that $x_{984} = x_{975} = x_{966} = x_{885} = x_{876} = 0$. And combining this with the second equation, we arrive at $12x_{993} = 210$, contradicting the fact that x_{993} is an integer. From this contradiction, we can conclude that no 4-cap exists which contains 21 points. It suffices now to verify that the following diagram represents a 4-cap with 20 points.



□

This clever method of counting pointed hyperplanes by triple hyperplane decompositions, despite serving us well thus far, has now reached the end of its tether. In fact, the problem of determining the size of a maximal 5-cap eluded mathematicians until as recently as 2002 when Edel, Ferret, Landjev and Storme [2] proved that a maximal 5-cap contains 45 points. Rather than elementary counting, the basis for their result was the Fourier transform, which assigns to a given function $f : \mathbb{Z}_3^d \to \mathbb{C}$, the function $\hat{f} : \mathbb{Z}_3^d \to \mathbb{C}$ defined by

$$\hat{f}(z) = \sum_{x \in \mathbb{Z}_3^d} f(x)\omega^{z \cdot x},$$

where $\omega = e^{\frac{2\pi i}{3}}$. In fact, their proof relied on the following amazing formula.

Theorem 4 *Let S be a subset of $\mathbb{Z}_3^d$ that contains P points and L lines. Then*

$$P + 6L = \frac{1}{3^d} \sum_{x \in \mathbb{Z}_3^d} [\hat{\chi}(x)]^3,$$

where $\hat{\chi}$ is the Fourier transform of the characteristic function[4] *for S.*

The theorem also provides a strong bound for the size of a 6-cap, although it gives progressively worse bounds for higher values of d. Amazingly enough, the size of a maximal 6-cap is still unknown, and the extent of our knowledge on the size of maximal d-caps is encapsulated in the following table.

d	1	2	3	4	5	6
size of a maximal d-cap	2	4	9	20	45	112–114

Problem: Show that you cannot have exactly 4 SETs among 7 cards in the original 4-dimensional version of SET.

[4] For a subset S of $\mathbb{Z}_3^d$, define the characteristic function $\chi : \mathbb{Z}_3^d \to \mathbb{C}$ by

$$\chi(x) = \begin{cases} 0 \text{ if } x \notin C \\ 1 \text{ if } x \in C \end{cases}$$

4 Final Thoughts

Despite being so intricately related to mathematics, the story of SET began in 1974 with epileptic dogs and a population geneticist by the name of Marsha Jean Falco. At the time, she was working on the problem of whether German Shepherds who had epilepsy might have inherited it. This research involved spotting patterns in large amounts of data. Thus, to make the job easier, Falco decided to represent blocks of information by coloured symbols drawn on small file cards. In a stroke of brilliance, she realised the potential for this to be the basis of a game and created the first ever SET deck with an engineering stencil and a deck of blank cards. For many years, the game of SET lay dormant to the world at large, only to be played by Falco and a small circle of family and friends. It was only in 1991, spurred on by her children, that she decided to reveal the game to the general public. Thus, SET was born.

Little did the creator of SET know that the game would have such deep connections with finite geometry and that the game would spawn so many mathematical problems. As we have seen, some of these have been solved with both elegant and complicated mathematics. However, many still remain unsolved, perhaps to be the mathematical fodder for future generations of mathematicians. The interested reader may like to try their hand at the following problem and to consult the article [1] for more SET theory.

It would be nice to know how large, in some sense, maximal caps can be. To this end, we can measure the size of a cap $C \subseteq \mathbb{Z}_3^d$ by its *solidity*, which is defined to be the number $\sqrt[d]{|C|}$. If $f(d)$ is the size of a maximal d-cap, note that $f(d) \geq 2^d$, since the points in $\mathbb{Z}_3^d$ whose coordinates are all 0 and 1 form a cap. Combined with the trivial bound $f(d) \leq 3^d$, we deduce that the solidity of a maximal cap is always in the interval $[2, 3]$.

(1) (Solved) Prove that $\lim_{d\to\infty} \sqrt[d]{f(d)}$ exists — this number is known as the *asymptotic solidity*.

(2) (Unsolved) It is known that the asymptotic solidity is greater than 2, but is it less than 3?

Let us come full circle and close the article with the very same quote with which we began.

> *Games are among the most interesting creations of the human mind, and the analysis of their structure is full of adventure and surprises. Unfortunately there is never a lack of mathematicians for the job of transforming delectable ingredients into a dish that tastes like a damp blanket.*

My hope is that I have been able to reveal some of the adventure and surprises in the mathematical analysis of SET without exposing too much of the damp blanket.

References

[1] B.L. Davis and D. Maclagan, *The card game SET!*, Math. Intelligencer **25** (2003), No 3.
[2] Y. Edel, S. Ferret, I. Landjev and L. Storme, *The classification of the largest caps in AG*(5, 3), J. Combin. Theory Ser. A **99** (2002), 95–110.
[3] J.R. Newman, *The World of Mathematics* (Simon and Schuster New York 1956).
[4] *The Set Game Company Homepage*, http://www.setgame.com.

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