Ramanujan and Fermat's Last Theorem

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Hardy [1] relates the following anecdote. "I remember going to see him [Ramanujan] when he was lying ill at Putney. I had ridden in taxi–cab No. 1729, and remarked that the number ($7 \times 13 \times 19$) seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.""

Indeed,

$$1729 = 9^3 + 10^3 = 12^3 + 1^3.$$

But there is another way in which this example is special. We know, since Euler, that the sum of two positive cubes is never a cube. But the above example shows that the sum of two positive cubes can do the next best thing – and that is, to miss a cube by as little as 1.

Indeed, Ramanujan left for us infinitely many examples of just that phenomenon. In his so-called "Lost Notebook" [4], he stated a result equivalent to the following.

If

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 791 \\ 812 \\ 1010 \end{pmatrix}, \quad \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 65601 \\ 67402 \\ 83802 \end{pmatrix}$$

and

$$\begin{pmatrix} x_{n+3} \\ y_{n+3} \\ z_{n+3} \end{pmatrix} = 82 \begin{pmatrix} x_{n+2} \\ y_{n+2} \\ z_{n+2} \end{pmatrix} + 82 \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} - \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \tag{1}$$

then

$$x_n^3 + y_n^3 = z_n^3 + (-1)^{n+1}.$$

In two articles in the Mathematics Magazine [2], [3] I gave two proofs of this amazing statement, and gave an explanation as to how Ramanujan may have obtained this result. I will give a brief exposition below.

Recently, I was inspired to guess that the vectors $\mathbf{x}_n = (x_n, y_n, z_n)^T$ might satisfy a different type of recurrence. Let me explain. The continued fraction for $\sqrt{2}$ is

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

If we cut this off after the nth 2, we obtain a rational close to $\sqrt{2}$, which we call the nth convergent to $\sqrt{2}$, and which we denote by $\frac{p_n}{q_n}$. Thus

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{n+2} \\ q_{n+2} \end{pmatrix} = 2 \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} + \begin{pmatrix} p_n \\ q_n \end{pmatrix}.$$

But it is also true that

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

and so

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Inspired by this, I guessed that there may exist a 3×3 matrix M such that the vectors $\mathbf{x}_n = (x_n, y_n, z_n)^T$ given by (1) satisfy $\mathbf{x}_{n+1} = M\mathbf{x}_n$ and $\mathbf{x}_n = M^n\mathbf{x}_0$.
And indeed there is!

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 63 & 104 & -68 \\ 64 & 104 & -67 \\ 80 & 131 & -85 \end{pmatrix}^n \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}. \tag{2}$$

(Incidentally, we can use Ramanujan's recurrence backwards as well as forwards to obtain triples (x_n, y_n, z_n), just as we can have n negative in (2).)

This is how one might discover Ramanujan's solutions.
Suppose you notice that

$$(x^2 + 16x - 21)^3 + (2x^2 - 4x + 42)^3$$

is even.
It follows that

$$(x^2 + 16x - 21)^3 + (2x^2 - 4x + 42)^3 = (x^2 - 16x - 21)^3 + (2x^2 + 4x + 42)^3.$$

Replace x by $2x + 1$ and divide by 64 to obtain

$$(x^2 + 9x - 1)^3 + (2x^2 + 10)^3 = (x^2 - 7x - 9)^3 + (2x^2 + 4x + 12)^3.$$

Replace x by v/u, multiply through by u^6 and rearrange to obtain

$$(9u^2 + 7uv - v^2)^3 + (10u^2 + 2v^3)^3 = (12u^2 + 4uv + 2v^2)^3 + (u^2 - 9uv - v^2)^3.$$

Now comes the Ramanujan-esque touch. Set $u = h_n, \quad v = h_{n-1}$ where the sequence $\{h_n\}$ is defined by

$$h_0 = 0, \quad h_1 = 1, \quad h_{n+2} = 9h_{n+1} + h_n \quad \text{for } n \geq 0.$$

This forces

$$u^2 - 9uv - v^2 = (-1)^{n+1}$$

and if we set

$$x_n = 9u^2 + 7uv - v^2, \quad y_n = 10u^2 + 2v^3, \quad z_n = 12u^2 + 4uv + 2v^2$$

then

$$x_n^3 + y_n^3 = z_n^3 + (-1)^{n+1}.$$

These are Ramanujan's x_n, y_n and z_n.

References

[1] G.H. Hardy, S. Ramanujan, *Collected papers of Srinivasa Ramanujan* (AMS Chelsea 2000), xxxv.
[2] M.D. Hirschhorn, *An amazing identity of Ramanujan*, Math. Mag. **3** (1995), 199–201.
[3] M.D. Hirschhorn, *A proof in the spirit of Zeilberger of an amazing identity of Ramanujan*, Math. Mag. **4** (1996), 267–269.
[4] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers* (Narosa Delhi 1988), 341.

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