

# Some Representations of $\pi$

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## Abstract

In this paper we consider a particular integral from which we may develop identities for  $\pi$ .

*Key words:* Hypergeometric summation, Definite integration.

*MSC:* Primary 33B15; Secondary 33C05.

## 1 Introduction

The ratio of the circumference to the diameter of a circle produces, arguably the most common (famous) mathematical constant known to the human race,  $\pi$ .

It appears that most school children know, or have been taught, that  $\pi$  has a value of about  $22/7$ . Throughout the ages  $\pi$  and its reciprocal has been represented by various formulae and the following are listed for interest.

Vieta ( $\sim 1579$ )

$$\frac{1}{\pi} = \frac{1}{2} \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}}} \cdots$$

J. Wallis ( $\sim 1650$ ) gave

$$\frac{\pi}{4} = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{(2k+1)^2} \right),$$

and

$$\frac{2}{\pi} = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{(2k)^2} \right).$$

Leibnitz ( $\sim 1670$ )

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Newton ( $\sim 1666$ )

$$\begin{aligned} \pi &= \frac{3\sqrt{3}}{4} + 24 \int_0^{1/4} \sqrt{x-x^2} dx \\ &= \frac{3\sqrt{3}}{4} + 2 - \frac{3}{4} \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{16^k (k+1)(2k+5)}. \end{aligned}$$

John Machin (1680–1751) found a formula, in 1706, of the form

$$\frac{\pi}{4} = 4 \arctan \left( \frac{1}{5} \right) - \arctan \left( \frac{1}{239} \right),$$

there are many other formulae of this form including  $\operatorname{arccot}(x)$  type functions, and the interested reader is referred to an excellent article by Wetherfield [9].

Fibonacci type formulae, see [3] for a fuller list of such formula, are given typically by

$$\frac{\pi}{2} = \sum_{k=1}^{\infty} \arctan\left(\frac{1}{F_{2k}}\right)$$

where  $F_{k+2} = F_{k+1} + F_k$ ,  $F_0 = 0$ ,  $F_1 = 1$ .

Euler ( $\sim 1748$ )

$$\pi^2 = 18 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}$$

Ramanujan (1914)

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \binom{2k}{k}^3 \frac{42k + 5}{2^{12k+4}}$$

Ramanujan gave many other identities for  $\pi$  and its reciprocal and the interested reader is referred to the excellent books by Berndt [4].

Comtet (1974)

$$\pi^4 = \frac{3240}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}$$

D. and G. Chudnovsky (1989)

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} (-1)^k \frac{(6k)!}{(k!)^3 (3k)!} \cdot \frac{13591409 + 545140134k}{(640320^3)^{k+\frac{1}{2}}}$$

Bailey, Borwein and Plouffe (1996)

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left[ \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right]. \tag{1}$$

Bellard [3]

$$\begin{aligned} \pi = \frac{1}{64} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{10k}} & \left[ \frac{1}{10k+9} - \frac{4}{10k+7} - \frac{4}{10k+5} \right. \\ & \left. - \frac{64}{10k+3} + \frac{256}{10k+1} - \frac{1}{4k+3} - \frac{32}{4k+1} \right]. \end{aligned}$$

Lupas [8]

$$\pi = 4 + \sum_{k=1}^{\infty} (-16)^k \frac{\binom{2k}{k} (40k^2 + 16k + 1)}{\binom{4k}{2k}^2 2k (4k + 1)^2}$$

The original Lupas formula contained a minor misprint which has been corrected here. Almkvist, Krattenthaler and Peterson [1]

$$\pi = \frac{1}{9 \cdot 25 \cdot 49} \sum_{k=0}^{\infty} \frac{-89286 + 3875948k - 34970134k^2 + 110202472k^3 - 115193600k^4}{\binom{8k}{4k} (-4)^k}$$

Borwein and Girgensohn [5]

$$\pi = \ln 4 + 10 \sum_{k=1}^{\infty} \frac{1}{2^k k \binom{3k}{k}}$$

Many other results of this type exist and recently Chudnovsky and Chudnovsky [6] obtained a master theorem from which they calculate

$$\frac{\pi}{2} = -1 + \sum_{r=1}^{\infty} \frac{2^r}{\binom{2r}{r}}$$

and using the Taylor series expansion of the  $\arcsin(x)$  function, we can obtain other similar formulae, such as

$$\pi = -3\sqrt{3} + \frac{9\sqrt{3}}{2} \sum_{r=1}^{\infty} \frac{r}{\binom{2r}{r}}.$$

In this paper we consider a general definite integral from which we can develop various other formulae for the representation of  $\pi$ . The following integral will be needed for the formulation of  $\pi$ .

## 2 The Integral

**Theorem 1** For  $k, m$  and  $\alpha$  real positive numbers and  $a \geq 1$ , then

$$I(a, k, m, \alpha) = \int_0^{1/a} \frac{x^m}{(1-x^k)^\alpha} dx \quad (2a)$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r! (rk+m+1) a^{rk+m+1}} \quad (2b)$$

$$= T_0 {}_2F_1 \left[ \begin{matrix} (m+1)/k, & \alpha \\ (m+1+k)/k \end{matrix} \middle| a^{-k} \right] \quad (2c)$$

where

$$T_0 = \frac{1}{(m+1)a^{m+1}}, \quad (3)$$

$(b)_s$  is Pochhammer's symbol defined by  $(b)_0 = 1$  and

$$(b)_s = b(b+1)\cdots(b+s-1) = \frac{\Gamma(b+s)}{\Gamma(b)},$$

$\Gamma(b)$  is the classical Gamma function and  ${}_2F_1$  is the Gauss hypergeometric function.

*Proof.*

$$\begin{aligned}
 I(a, k, m, \alpha) &= \int_0^{1/a} \frac{x^m}{(1-x^k)^\alpha} dx \\
 &= \int_0^{1/a} x^m \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} x^{kr} dx \\
 &= \int_0^{1/a} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} x^{kr+m} dx \\
 &= \left[ \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} \frac{x^{kr+m+1}}{(kr+m+1)} \right]_0^{1/a} \\
 &= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r! (kr+m+1) a^{kr+m+1}} \\
 &= T_0 {}_2F_1 \left[ \begin{matrix} (m+1)/k, & \alpha \\ (m+1+k)/k \end{matrix} \middle| a^{-k} \right]
 \end{aligned}$$

where  $T_0$  is given by (3). □

We can now match (2b) and (2c) so that

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r! (rk+m+1) a^{rk+m+1}} = T_0 {}_2F_1 \left[ \begin{matrix} (m+1)/k, & \alpha \\ (m+1+k)/k \end{matrix} \middle| a^{-k} \right], \tag{4}$$

and the left hand side of (4) converges absolutely for  $|a^{-k}| < 1$ . In the case when  $a = 1$  we can state the following corollary.

**Corollary 1** For  $a = 1, k > 0, m > -1$  and  $0 < \alpha < 1$  then

$$\pi = \frac{k \Gamma(\alpha) \Gamma(1-\alpha + (1+m)/k) \sin(\alpha\pi)}{\Gamma((1+m)/k)} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r! (rk+m+1)}. \tag{5}$$

*Proof.* From (2a)

$$I(1, k, m, \alpha) = \int_0^1 \frac{x^m}{(1-x^k)^\alpha} dx$$

and by the substitution  $z = 1 - x^k$  we have

$$I(1, k, m, \alpha) = \frac{1}{k} B \left( 1 - \alpha, \frac{1+m}{k} \right) \tag{6}$$

where  $B$  is the classical Beta function. From (2b) and (6)

$$\begin{aligned}
 B \left( 1 - \alpha, \frac{1+m}{k} \right) &= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r! (rk+m+1)} \\
 \frac{\Gamma(1-\alpha) \Gamma((1+m)/k)}{\Gamma(1-\alpha + (1+m)/k)} &= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r! (rk+m+1)}.
 \end{aligned}$$

From

$$\Gamma(1-\alpha) = \frac{\pi \operatorname{cosec}(\alpha\pi)}{\Gamma(\alpha)},$$

then

$$\frac{\pi \operatorname{cosec}(\alpha\pi) \Gamma((1+m)/k)}{\Gamma(\alpha) \Gamma(1-\alpha+(1+m)/k)} = k \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!(rk+m+1)} \tag{7}$$

and a rearrangement leads to the result (5). □

*Remark* Bailey, Borwein, Borwein and Plouffe [2] utilised (2a) for  $a = \sqrt{2}$ ,  $\alpha = 1$ ,  $k = 8$  and  $m = \beta - 1$ ,  $\beta = 0, 1, \dots, 7$  to prove the new formula (1). Subsequently Hirschhorn [7] has shown that (1) can be obtained from standard integration procedures.

Some examples are now given.

*Example 1* For  $m + 1 = 3k/2$ , then

$$\pi = \frac{\Gamma(\alpha) \Gamma(5/2 - \alpha) \sin(\alpha\pi)}{\Gamma(3/2)} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!(r+3/2)}.$$

For  $\alpha = 1/2$ , and using

$$2^{2r} \frac{(1/2)_r}{r!} = \binom{2r}{r}$$

we have

$$\frac{\pi}{4} = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{1}{4^r(2r+3)}.$$

In general, from (5), for  $\alpha = 1/2$ , we can deduce, after some basic algebra, that

$$\pi = \frac{2p!}{(1/2)_p} \sum_{r=0}^{\infty} \binom{2r}{r} \frac{1}{4^r(2r+2p+1)}, \quad p = 0, 1, 2, \dots$$

and

$$\frac{(p-1)!}{(1/2)_p} = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{1}{4^r(r+p)}, \quad p = 1, 2, 3, \dots,$$

which is a special case of the Gauss sum.

*Example 2* For  $m + 1 = k(\alpha + p)$ ,  $p = 0, 1, 2, \dots$  and  $\alpha = 7/30$

$$\pi = \frac{15p! \left( \sqrt{30+6\sqrt{5}} + 1 - \sqrt{5} \right)}{4(7/30)_p} \sum_{r=0}^{\infty} \frac{(7/30)_r}{r!(30r+30p+7)}.$$

For  $m = 25$ ,  $k = 24$ ,  $\alpha = 1/12$

$$\pi = 36 \left( \sqrt{6} - \sqrt{2} \right) \sum_{r=0}^{\infty} \frac{(1/12)_r}{r!(12r+13)}.$$

Both of these nice representations for  $\pi$ , it can be argued, are less suitable for the numerical calculation of  $\pi$  than those given in the previous example.

*Example 3* In the case when  $(m + 1)/k = \text{integer} = s$  say, and using (7), then we obtain the numerical constant

$$B(s, 1 - \alpha) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!(r+s)}.$$

For other cases of the value of  $a$  in the integral (2a) we may also obtain identities for  $\pi$ . In these cases the integral is a little more difficult to handle and these results will be reported in another forum. As an indication of the type of results we can derive, for an arbitrary choice of  $(a, k, m, \alpha) = (2, 2, 120, 9/2)$  we obtain

$$\pi = \frac{\Omega_1}{\Omega_2 \sqrt{3}} + \frac{1}{\Omega_3} \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(2r+1)(2r+3)(2r+5)(2r+7)}{(2r+121)16^r},$$

where

$$\begin{aligned} \Omega_1 &= 15604102274295581508678435968572864501995513795052733 \\ &= (46042305118509401202197)(338907929004245243145594887689), \\ \Omega_2 &= 2^6 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot \\ &\quad 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113 \end{aligned}$$

and

$$\Omega_3 = 2^{11} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113.$$

There are, of course, a multitude in this class of identities for  $\pi$  for other various values of  $(a, k, m, \alpha)$ .

## Acknowledgement

I would like to thank an anonymous referee for some useful and helpful comments on this paper.

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Received 7 May 2003, accepted 26 February 2004.