1 Solving linear first-order ordinary differential equations

Clarify the purpose of the class: to use the integrating factor method to solve a linear first-order ordinary differential equation.

Identify key concepts that need to be conveyed:

1. A *differential equation* is any equation involving a derivative

$$\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = -y$$

$$y' + p(x)\, y = q(x)$$

$$\left(\frac{dy}{dx}\right)^2 + e^x \frac{dy}{dx} - 3y = 2x + 1$$

$$f'(x) = (f(x))^2$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y^2 = \cosh x$$

$$\frac{\partial y}{\partial s} + \frac{\partial y}{\partial t} = s^2 + t^2$$

2. An *ordinary* differential equation is a differential equation not involving

partial derivatives.

$$\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = -y$$

$$y' + p\left(x\right)y = q\left(x\right)$$

$$\left(\frac{dy}{dx}\right)^2 + e^x\frac{dy}{dx} - 3y = 2x + 1$$

$$f'\left(x\right) = \left(f\left(x\right)\right)^2$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y^2 = \cosh x$$

3. A *first-order* differential equation is any differential equation involving a derivative (but not second or other higher order derivatives). For example

$$\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = -y$$

$$y' + p(x)\, y = q(x)$$

$$\left(\frac{dy}{dx}\right)^2 + e^x \frac{dy}{dx} - 3y = 2x + 1$$

$$f'(x) = (f(x))^2$$

are all first-order differential equations.

4. A *linear* differential equation has only linear combinations of its derivatives, from zero order up.

$$\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = -y$$

$$y' + p(x)\, y = q(x)$$

In all the above we are solving for an unknown function y or f.

1.1 Identify the background knowledge necessary to understand the above concepts.

Students will need to be very competent in differentiation, have reasonable integration skills, know what it means to solve an equation, and also (more difficult) to solve a functional equation.

1.2 Notation

You are all used to notations for function, namely for

$$y = f(x)$$

we refer to the function as f or $f(x)$ or y, or $y(x)$ to emphasise the dependence of y on x. There are correspondingly different notations used for ordinary derivatives:

$$\frac{dy}{dx}, y', y'(x), \dot{y}, f'(x), f'$$

which should be discussed.

1.3 Some basic equations

At this stage, it would be useful to consider the most basic ordinary linear first-order differential equation, typified by the example

$$\frac{dy}{dx} = \cos x \tag{1}$$

The students need to understand that the aim here is to find a function $y(x)$ (and we write it this way to emphasise the dependence of y on x) that satisfies the given equation. The students should have the background to do this one by inspection. It is clear that $y = \sin x$ satisfies the equation, as does $y = \sin x + 10$. Perhaps with these two solutions the students will begin to see the connection with integration; the solution is

$$y = \int \cos x \, dx = \sin x + C, \text{ where } C \text{ is a constant.}$$

This example also alerts the student that there isn't a unique solution.

We now progress to the slightly more complicated equation

$$\frac{dy}{dx} = -y \tag{2}$$

(compare this with equation (1)) which we can also write as

$$\frac{dy}{dx} = -y(x) \text{ or } f'(x) = -f(x)$$

Such equations are very important in growth/decay models of populations and radioactive materials. Now it looks like that solving a differential equation is tied up with working out an integral, but that isn't all there is to the problem. We essentially solved the equation (1) by integrating both sides of the equation with respect to x.

$$y = \int dy = \int \frac{dy}{dx} dx = \int \cos x \, dx = \sin x + C$$

(This *extra* Leibniz notation needs to be discussed at some stage; treat $\frac{dy}{dx}$ as a fraction and it will all work out.) If we attempt to take the same approach to solving (2) then we obtain

$$y = \int dy = \int \frac{dy}{dx} dx = -\int y \, dx$$

(remember that $y = y(x)$ is a function of x), but we haven't made any progress as the resulting equation

$$y = -\int y\,dx \tag{3}$$

is just as complicated. Our original equation (2) has the unknown variable y on both sides and involves the derivative $\frac{dy}{dx}$, and our new equation (3) also has the unknown variable y on both sides but now involves the integral $\int y\,dx$. So all that we have succeeded in doing is to replace our differential equation by an integral equation.

At this stage there needs to be a discussion about what might work, including agreement that there doesn't seem to be anyway to avoid integration (how else can we take care of the derivative?). I am now going to pull a (mathematical) rabbit out of a hat, by multiplying both sides of (2) by e^x, to obtain

$$\frac{dy}{dx}e^x = -ye^x \iff \frac{dy}{dx}e^x + ye^x = 0$$

and then observing that the resulting equation is just

$$\frac{d}{dt}\left(ye^x\right) = 0$$

using the product rule for differentiation. Then integrating both sides of the equation gives

$$ye^x = \int \frac{d}{dt}\left(ye^x\right)\,dx = C, \text{ where } C \text{ is a constant}$$

and the solution is just

$$y = Ce^{-x}$$

There are four important points to be made.

1. Integration is a particular case of solving a differential equation.
2. The solution isn't unique as there is always a constant of integration.
3. The product rule for differentiation seems to get over the problem of bringing together the mixed sum involving the function and its derivative.
4. The solution can be checked by substituting it back into the equation (2).

1.4 Integrating factor

A natural question is how e^x was chosen. This can be approached in a couple of ways. The first is to start discussing the general equation and just write down what we are going to call the *integrating factor*. Any linear first-order ordinary differential equation can be written in the form

$$a(x)\,\frac{dy}{dx} + b(x)\,y = c(x)$$

where $a(x)$ is a non-zero function (the trivial case can also be discussed), and multiplying through by $a(x)^{-1}$ we obtain

$$\frac{dy}{dx} + a(x)^{-1} b(x) y = a(x)^{-1} c(x)$$

and then, replacing $a(x)^{-1} b(x)$ by $p(x)$ and $a(x)^{-1} c(x)$ by $q(x)$,

$$\frac{dy}{dx} + p(x) y = q(x)$$

Equation $\frac{dy}{dx} + p(x) y = q(x)$ is referred to as being in *standard form.* Make special note of the importance of the plus sign. For example, the equation

$$\frac{dy}{dx} - 3y = \sin x$$

is not in standard form, but it can (and should) be rewritten as

$$\frac{dy}{dx} + (-3) y = \sin x \tag{4}$$

so that in this case $p(x) = -3$ and $q(x) = \sin x$.

We are now going to solve

$$\frac{dy}{dx} + p(x) y = q(x)$$

by multiplying each side by

$$e^{\int p(x) dx}$$

which is called the *integrating factor*, to obtain

$$\frac{dy}{dx} e^{\int p(x) dx} + y p(x) e^{\int p(x) dx} = q(x) e^{\int p(x) dx}$$

For our equation (4) the integrating factor is

$$e^{\int (-3)\, dx} = e^{-3x}$$

(there will also need to be some discussion around why the constant of integration isn't needed), and (4) becomes

$$\frac{dy}{dx} e^{-3x} + y(-3) e^{-3x} = e^{-3x} \sin x$$

which can be rewritten as (remember the product rule for differentiation)

$$\frac{d}{dx}\left(y e^{-3x}\right) = e^{-3x} \sin x$$

Integrating both sides gives

$$y e^{-3x} = \int e^{-3x} \sin x \, dx = -\frac{1}{10} e^{-3x} (\cos x + 3 \sin x) + C$$

and solving for y gives

$$y = -\frac{1}{10}(\cos x + 3\sin x) + Ce^{3x}$$

We can now make three observations.

1. There is only one (up to a constant multiple) integrating factor that will work. (Explain what 'will work' means and refer to the pitfall below.)
2. There is a constant that appears as a multiplier rather than just added to the final result, in contrast to what happens with straight integration.
3. The solution can be checked by substituting it back into the equation (4).

Whether we then go through the process of justifying this particular integrating factor depends on the class.

1.5 A possible pitfall

We could now look at the standard mistake made by students, that of overlooking the minus sign and writing the integrating factor as

$$e^{\int 3\,dx} = e^{3x}$$

The problem now is that multiplying by e^{3x} will **not** lead to a simplification of

$$\frac{dy}{dx}e^{3x} - y3e^{3x}$$

as the (total) derivative of a product term.

Finally emphasise that there is no change to the approach using different notation for the *independent* variable (x, t) on which the variable y *depends*.