Young tableaux

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Partitions and Young diagrams

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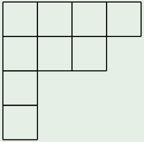
Example

$(4, 3, 1, 1)$ is a partition of 9. The Young diagram of it is



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Let f^λ denote the number of SYT of shape λ.

Hook-length formula

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The hook-length $h_\lambda(b)$ of a box b in a Young diagram λ is the number of boxes directly to its left, or directly below it, including b itself.

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Theorem (Frame-Robinson-Thrall)

$$f^\lambda = \frac{n!}{\prod_b h_\lambda(b)}$$

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Theorem (Frame-Robinson-Thrall)

$$f^\lambda = \frac{n!}{\prod_b h_\lambda(b)}$$

Thus

$$f^{(4311)} = \frac{9!}{7.4.3.1.5.2.1.2.1} = 216$$

More identities

Theorem (Frobenius identity)

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as λ varies over partitions of n.

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Theorem

$$\sum_{\lambda} f^{\lambda} = \#\{w \in S_n \mid w^2 = 1\}$$

as λ varies over partitions of n.

$n = 4$

1	2	3	4

1	2	3
4		

1	2	4
3		

1	3	4
2		

1	2
3	4

1	3
2	4

1	2
3	
4	

1	3
2	
4	

1	4
2	
3	

1
2
3
4

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 24$$

Stanley's $2^{\lfloor n/2 \rfloor}$ conjecture

Define the sign $\mathrm{sign}(T)$ of a SYT by

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 6 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$$

$$r(T) = 134725968$$

$$\mathrm{sign}(T) = (-1)^{\#\{(3,2),(4,2),(7,2),(7,5),(7,6),(9,6),(9,8)\}} = -1$$

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Theorem (L.)

$$\sum_T \mathrm{sign}(T) = 2^{\lfloor n/2 \rfloor}$$

where the sum is over all SYT T with n boxes.

$n = 4$

T	$r(T)$	$\mathrm{sign}(T)$
1 2 3 4	1234	1
1 2 3 4	1234	1
1 2 4 3	1243	-1
1 3 4 2	1342	1
1 2 3 4	1234	1
1 3 2 4	1324	-1

1 2 3 4	1234	1
1 3 2 4	1324	-1
1 4 2 3	1423	1
1 2 3 4	1234	1

An open problem

$$\sum_T 1 = \#\{w \in S_n \mid w^2 = 1\}$$

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Problem: What happens if you replace $\mathrm{sign}(T)$ by other functions of $r(T)$ (or T)?

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For example, the irreducible characters of S_n.

Littlewood-Richardson numbers

Let λ, μ, ν be partitions. The Littlewood-Richardson number $c^{\lambda}_{\mu\nu}$ is the number of SYT of shape λ/ν such that when you slide boxes in you get

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$\lambda = (4, 3, 2)$, $\mu = (3, 2, 1)$, $\nu = (2, 1)$

Then $c^{\lambda}_{\mu\nu}$ counts the following two tableau

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$$\begin{array}{cccc} \color{red}{1} & & 2 & 3 \\ 4 & 5 & & \\ 6 & & & \end{array}$$

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Horn's Theorem

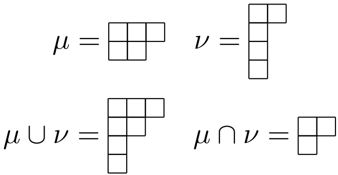
Theorem (Horn, Klyachko, Knutson-Tao)

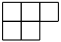
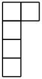
A $p \times p$ Hermitian matrix C can be expressed as $C = A + B$ where A, B are Hermitian matrices with eigenvalues $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_p)$ and $\beta = (\beta_1 \geq \cdots \geq \beta_p)$ if and only if the eigenvalues $\gamma = (\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p)$ of C satisfy

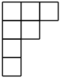
1. $$\sum_{i=1}^{p} \gamma_i = \sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{p} \beta_i$$
2. $$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

for every triple (I, J, K) "corresponding" to $c_{\mu\nu}^{\lambda} \neq 0$.

Inequalities for Littlewood-Richardson numbers



$\mu =$  $\nu =$ 

$\mu \cup \nu =$  $\mu \cap \nu =$ 

Theorem (L.-Postnikov-Pylyavskyy)

$$c_{\mu\nu}^{\lambda} \leq c_{\mu\cup\nu,\mu\cap\nu}^{\lambda}$$

for each λ.

This proved a conjecture of Okounkov ($c_{\mu\nu}^{\lambda} \leq c_{(\mu+\nu)/2,(\mu+\nu)/2}^{\lambda}$) concerning log-concavity of characters, and a conjecture of Fomin-Fulton-Li-Poon, arising from the study of singular values of (submatrices of) Hermitian matrices.

Random Young tableau

What is the shape of a typical SYT?

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Put the measure $M(\lambda) = \frac{(f^\lambda)^2}{n!}$ on shapes of size n. So, one has

$$M(\ydiagram{1,1,1,1}) = \frac{1}{24} \quad M(\ydiagram{2,1,1}) = \frac{9}{24} \quad M(\ydiagram{2,2}) = \frac{4}{24} \quad M(\ydiagram{3,1}) = \frac{9}{24} \quad M(\ydiagram{4}) = \frac{1}{24}$$

The (Frobenius identity) says that this is a probability measure.

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Let $E(n)$ be the expected length of the first row of a shape with n boxes. So

$$E(4) = \frac{1}{24} + 2.\frac{9}{24} + 2.\frac{4}{24} + 3.\frac{9}{24} + 4.\frac{1}{24} = 2.416...$$

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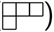
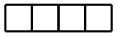
$M($$) = \frac{1}{24}$ $M($$) = \frac{9}{24}$ $M($$) = \frac{4}{24}$ $M($$) = \frac{9}{24}$ $M($$) = \frac{1}{24}$

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Theorem (Logan-Shepp, Vershik-Kerov)

$$\lim_{n\to\infty} \frac{E(n)}{\sqrt{n}} = 2$$

Random tableaux and random matrices

Theorem (Baik-Deift-Johansson, 1998)

The random variables

1. *"length of first row of random partition" and*
2. *"largest eigenvalue of a random $n \times n$ Hermitian matrix"*

converge (suitably normalized) to the same distribution as $n \to \infty$.

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Theorem (Okounkov, Borodin-Okounkov-Olshanski, Johansson)

Same holds for

1. *joint distribution of the first k rows of a random partition, and*
2. *largest k eigenvalues of a random $n \times n$ Hermitian matrix*

as $n \to \infty$.