AustMS ECR Workshop
Adelaide, 29092009

Three combinatorial dual identities

and their proximity to something useful

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Funded by an ARC Discovery Grant

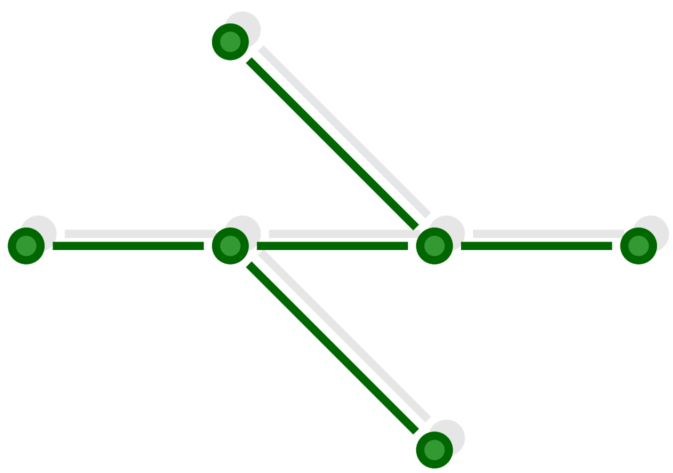
G.H. Hardy
(1877-1947)

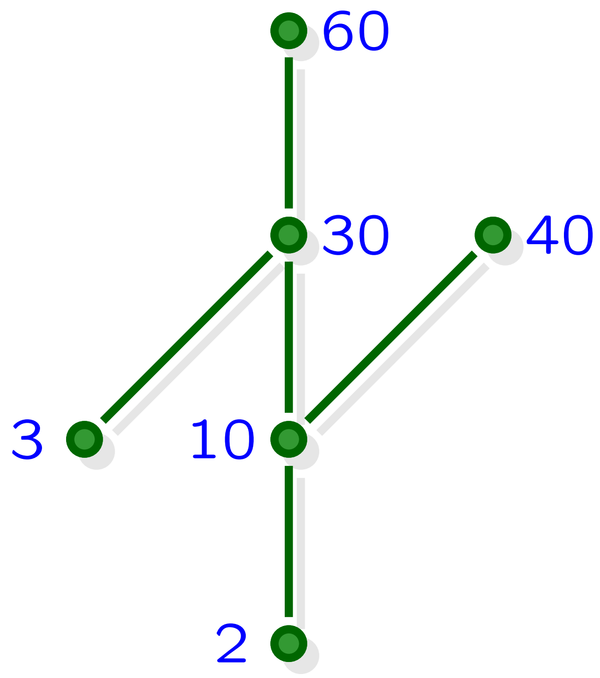
*“I have never done anything ‘useful’.
No discovery of mine has made, or is likely to make,
directly or indirectly, for good or ill,
the least difference to the amenity of the world.”*

*“Mathematicians may be justified in rejoicing
that there is one science at any rate, and that their own,
whose very remoteness from ordinary human activities
should keep it gentle and clean”*

Duality Theorem 1:

Greene's Theorem

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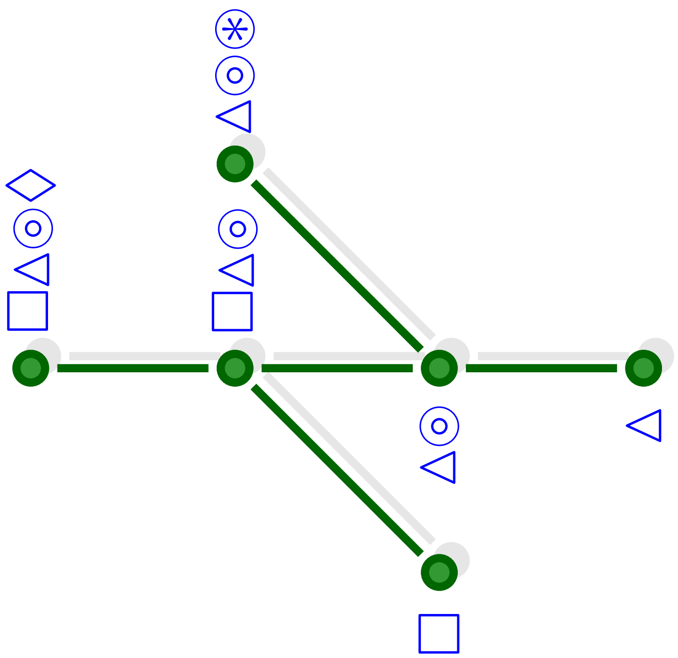
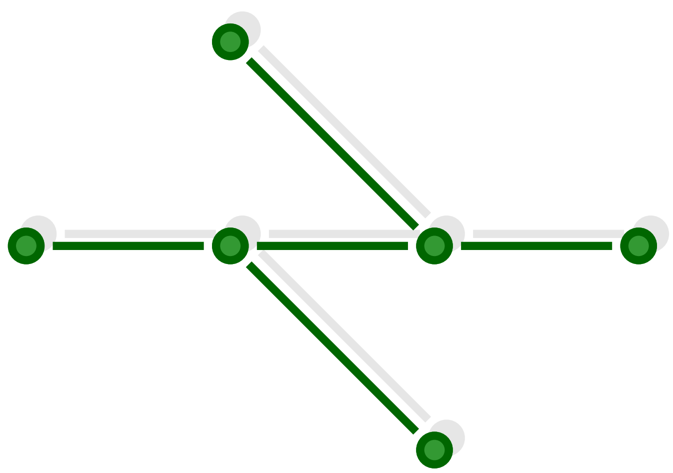
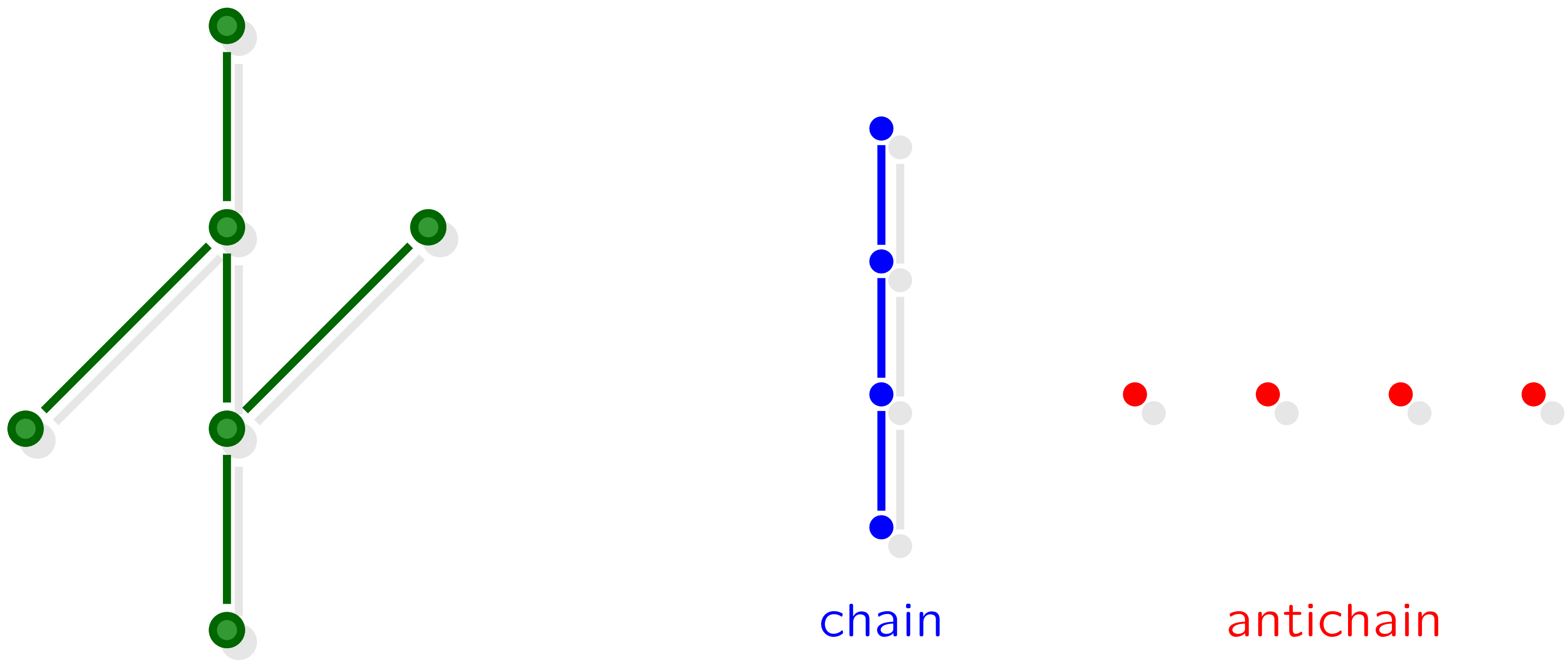
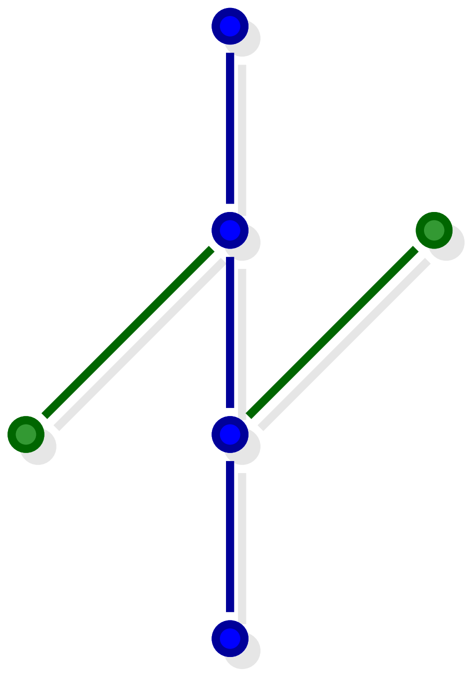
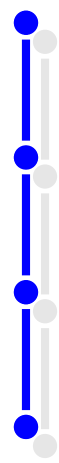




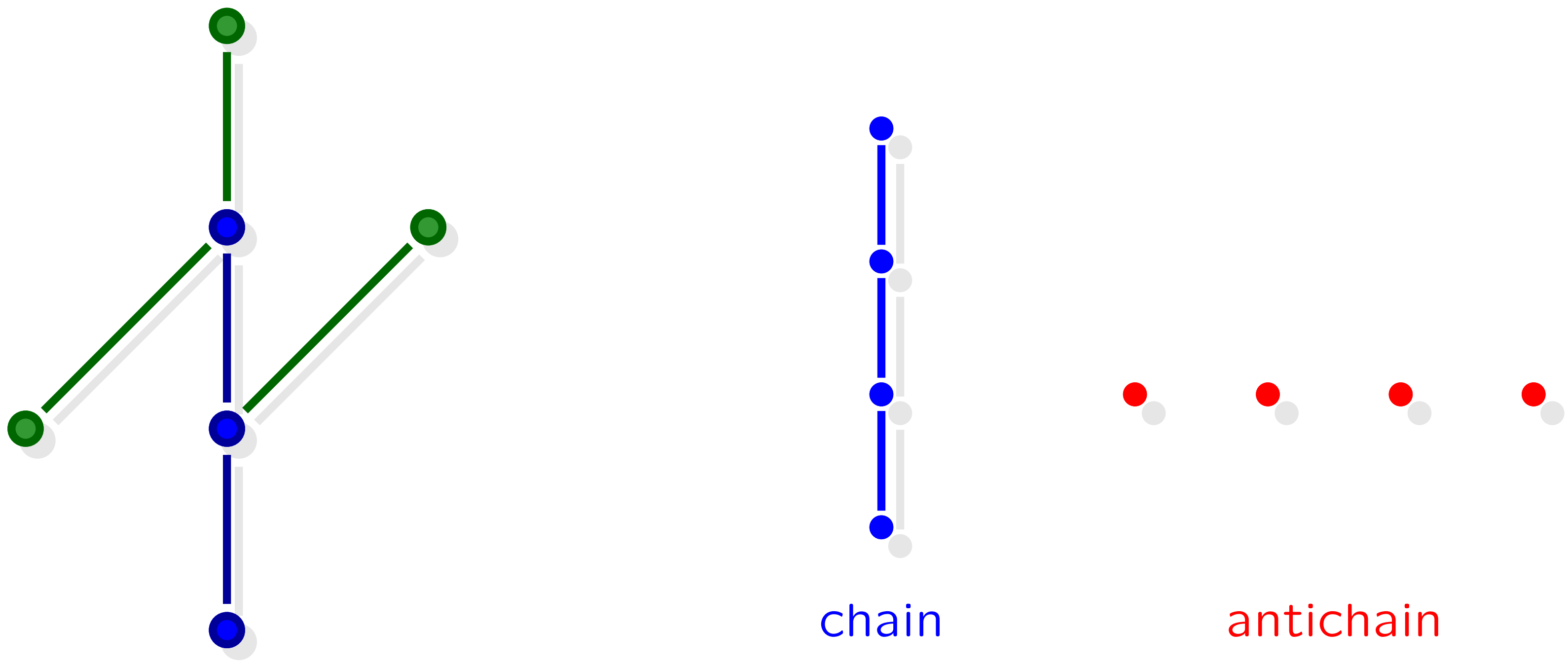

chain
antichain





chain

antichain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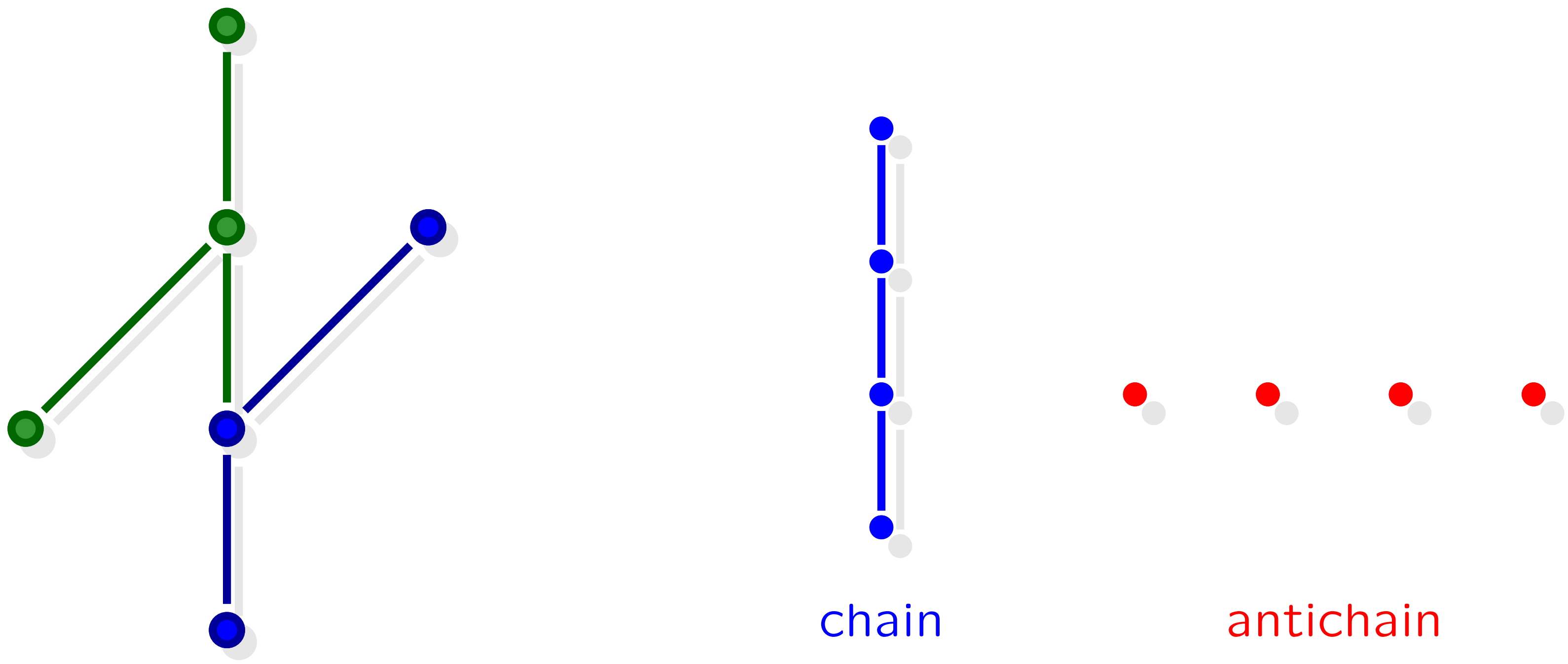
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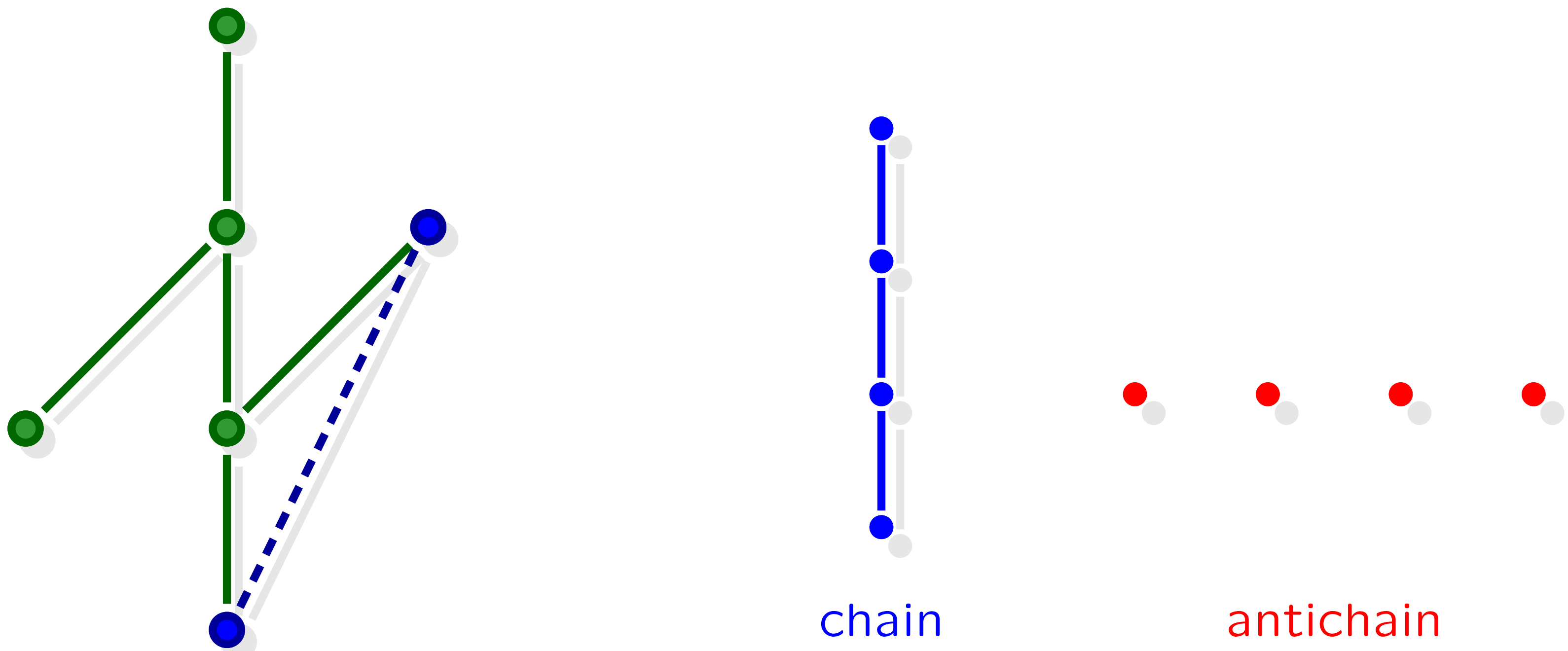
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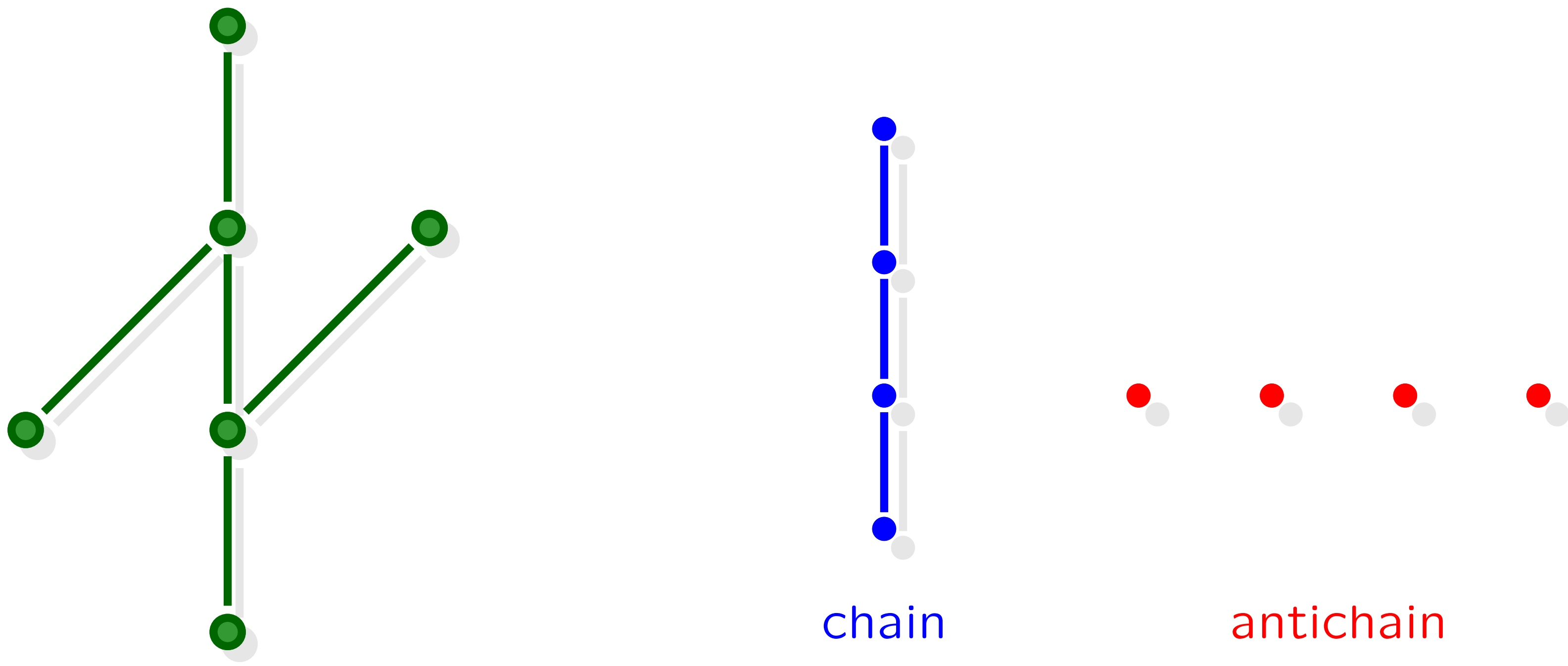
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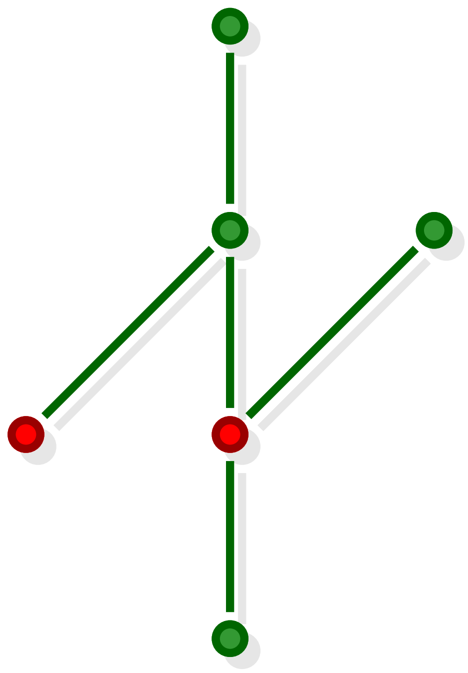
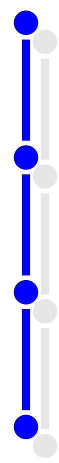
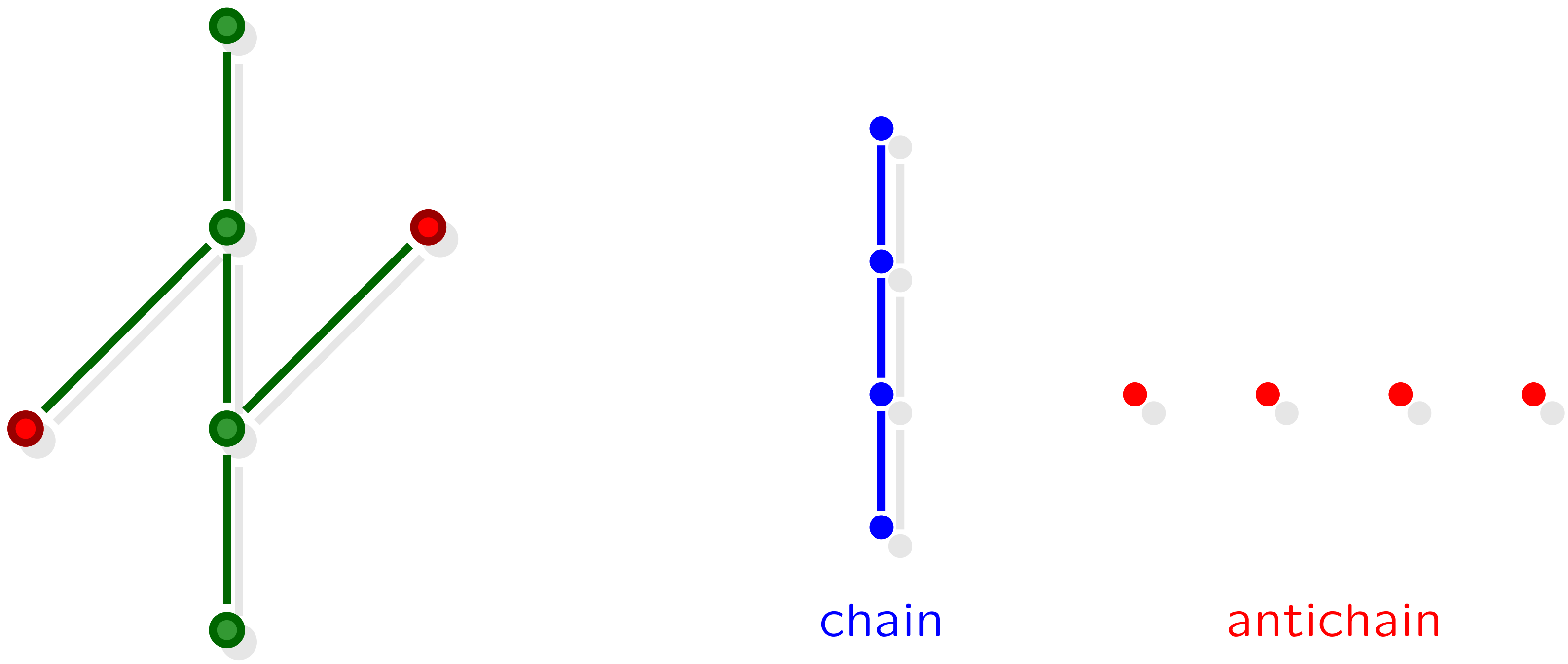




chain

antichain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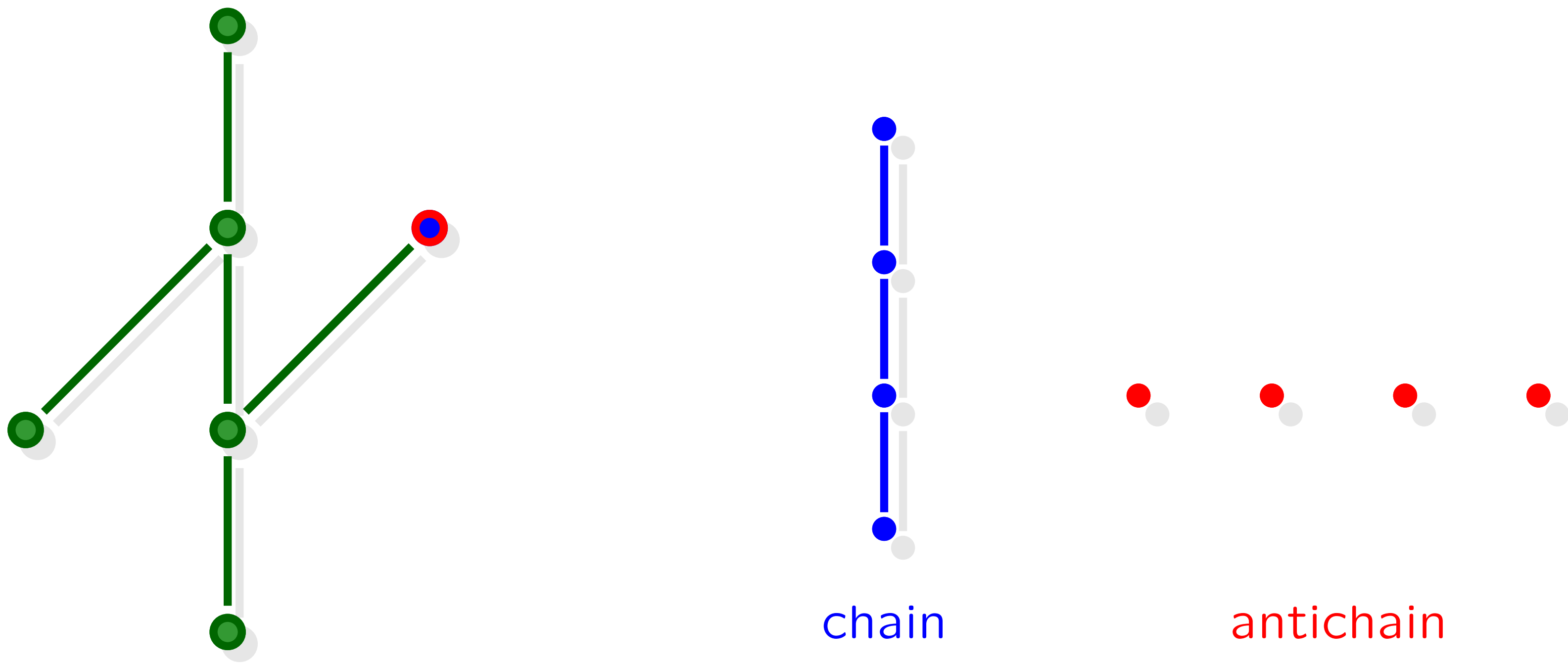
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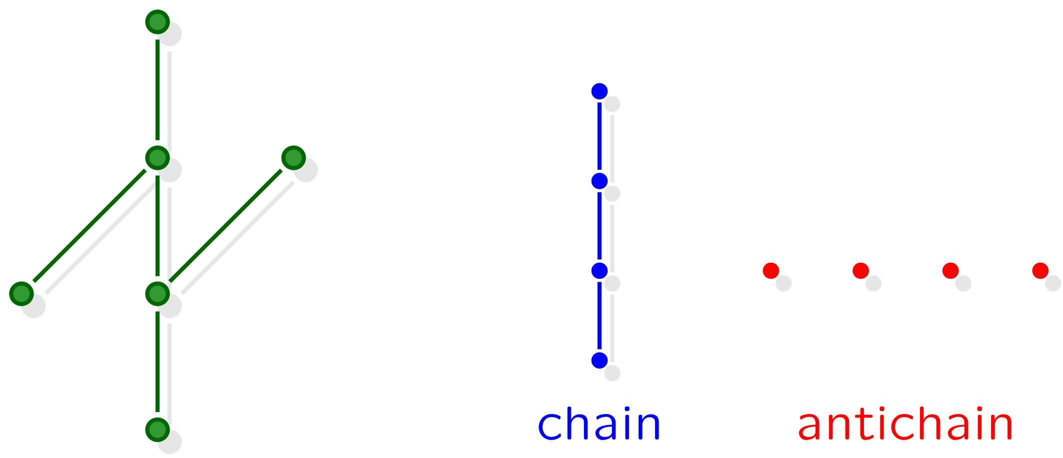
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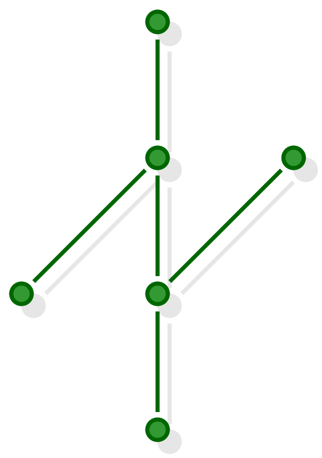
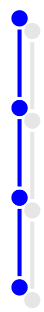
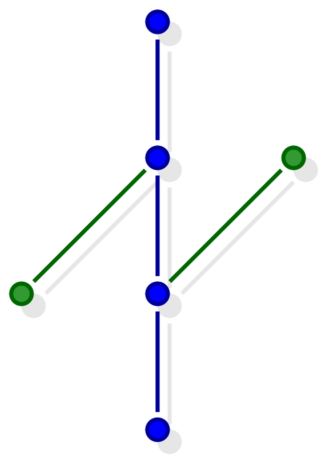


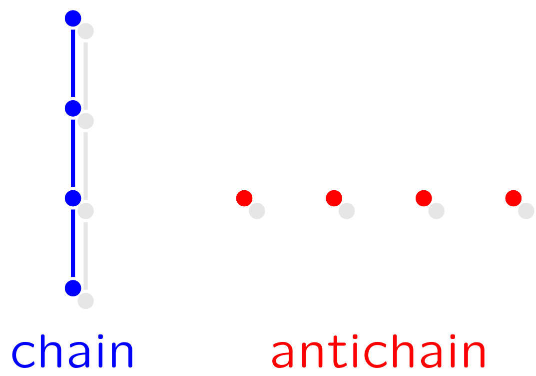


chain

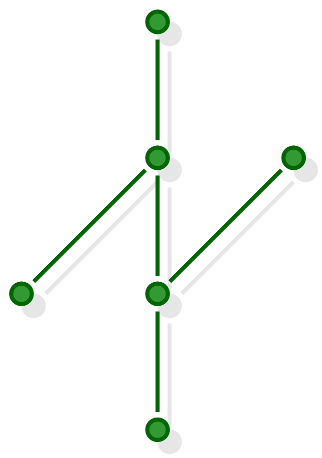
antichain

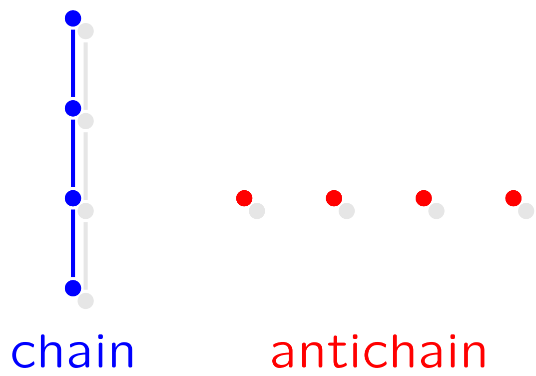
c_1 = maximal size of 1 chain =





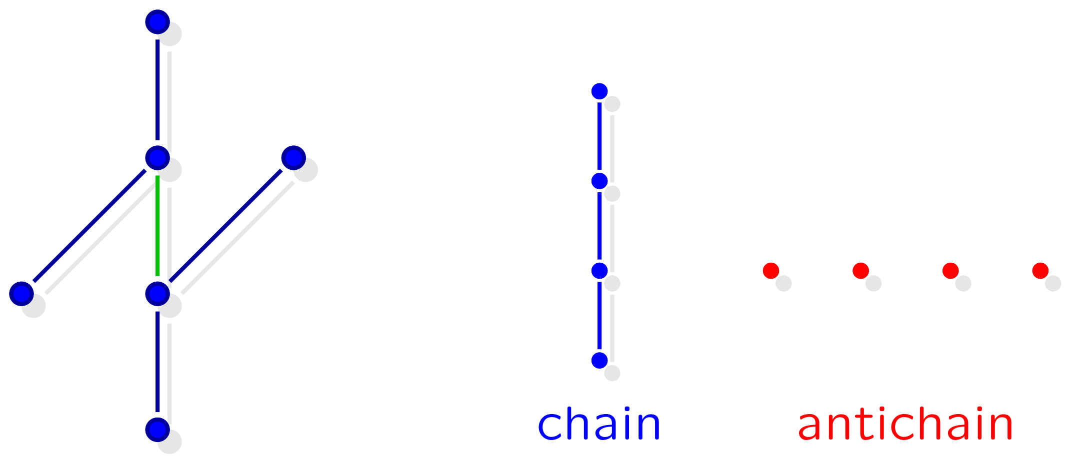
c_1 = maximal size of 1 chain = 4





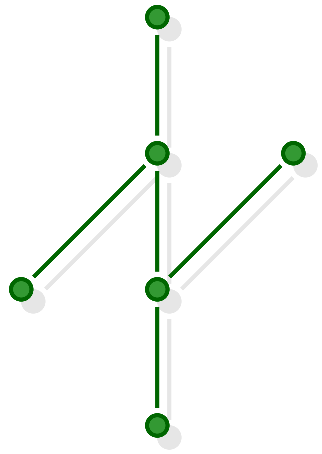
c_1 = maximal size of 1 chain = 4

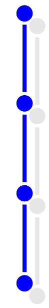
c_2 = maximal size of 2 chains =



c_1 = maximal size of 1 chain = 4

c_2 = maximal size of 2 chains = 6





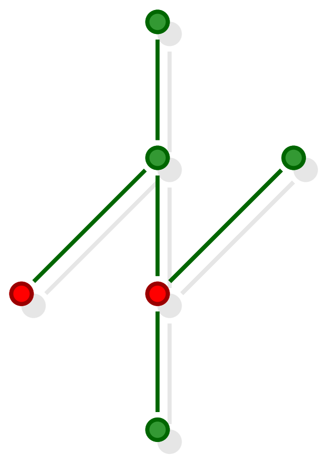
chain

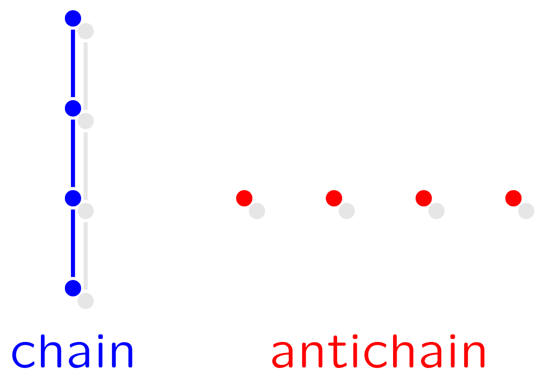
antichain

c_1 = maximal size of 1 chain = 4

c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain =

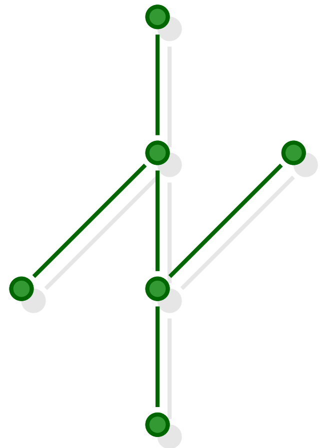


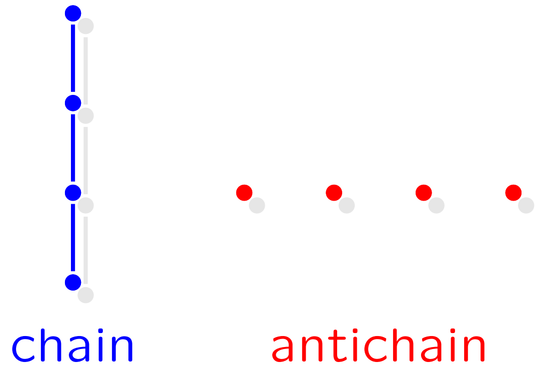


c_1 = maximal size of 1 chain = 4

c_2 = maximal size of 2 chains = 6

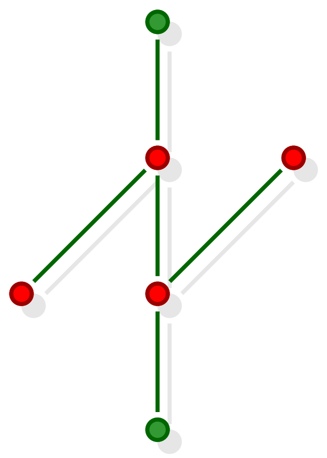
a_1 = maximal size of 1 antichain = 2

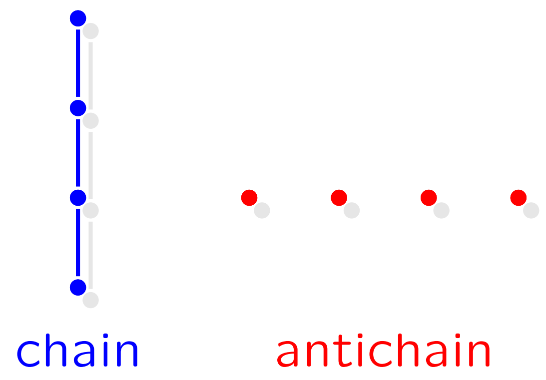




c_1 = maximal size of 1 chain = 4
c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2
a_2 = maximal size of 2 antichains =



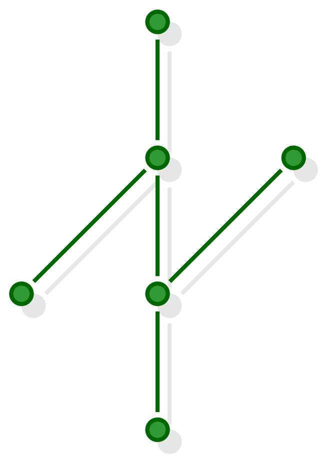


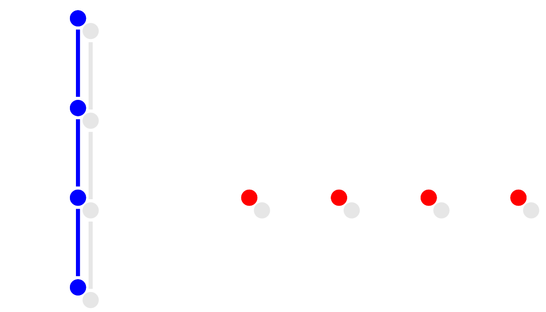
c_1 = maximal size of 1 chain = 4

c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2

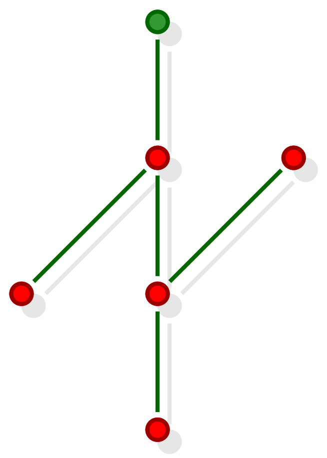
a_2 = maximal size of 2 antichains = 4

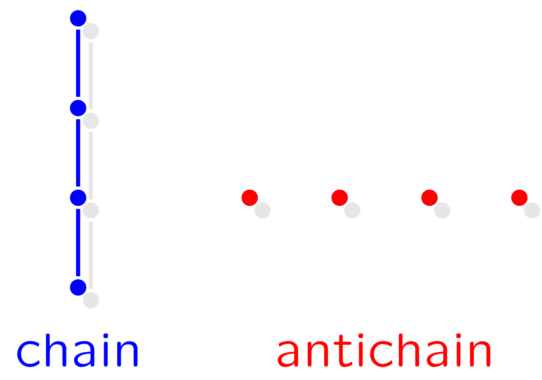




c_1 = maximal size of 1 chain = 4
c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2
a_2 = maximal size of 2 antichains = 4
a_3 = maximal size of 3 antichains =





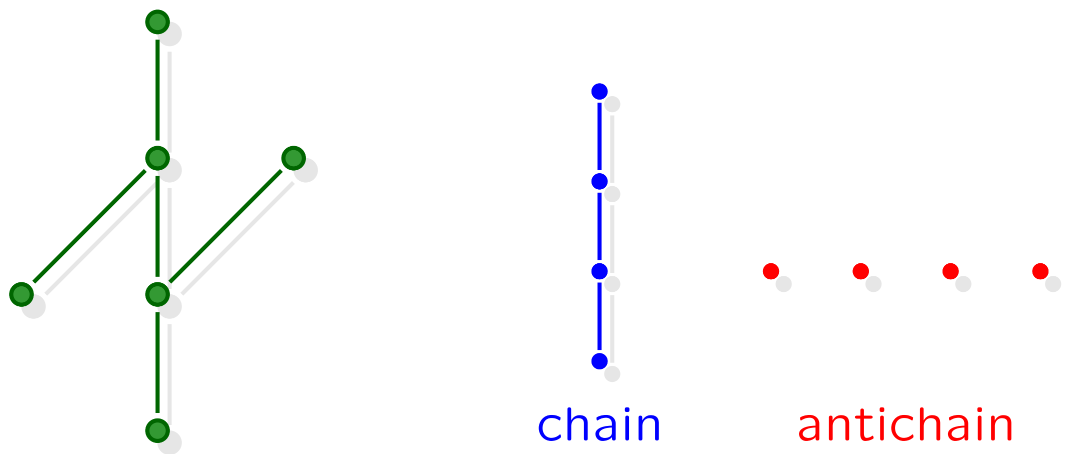
c_1 = maximal size of 1 chain = 4

c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2

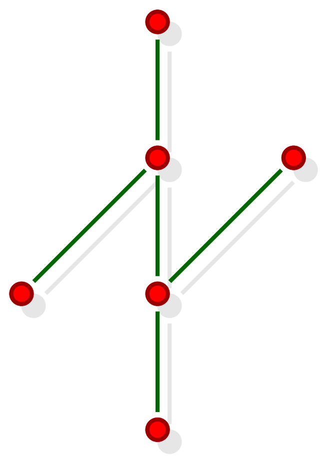
a_2 = maximal size of 2 antichains = 4

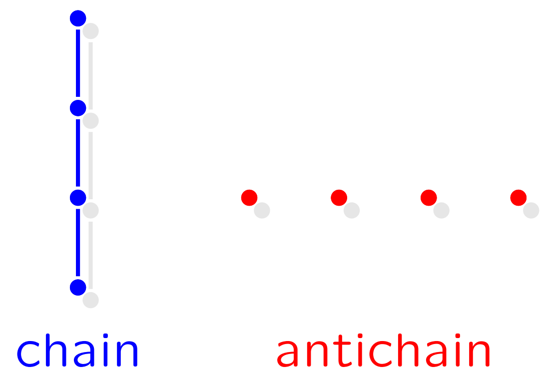
a_3 = maximal size of 3 antichains = 5



c_1 = maximal size of 1 chain = 4
c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2
a_2 = maximal size of 2 antichains = 4
a_3 = maximal size of 3 antichains = 5
a_4 = maximal size of 4 antichains =





c_1 = maximal size of 1 chain = 4

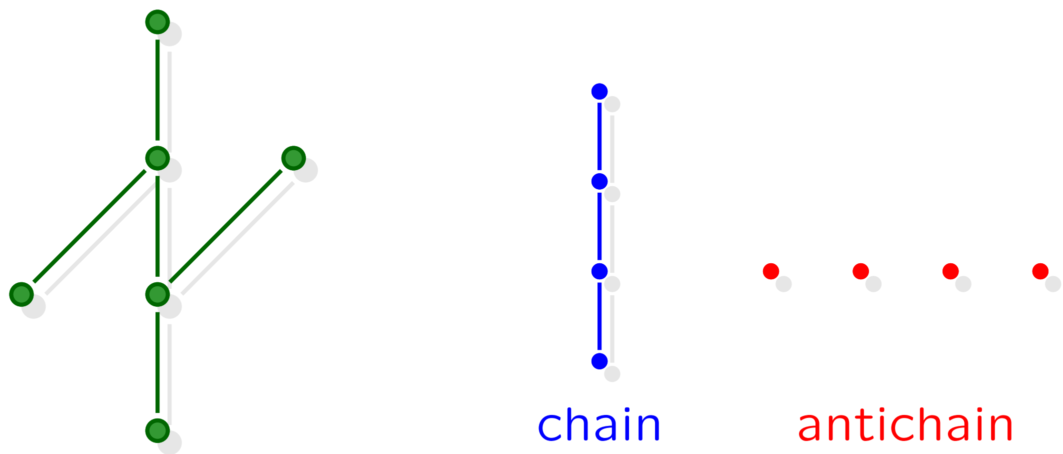
c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2

a_2 = maximal size of 2 antichains = 4

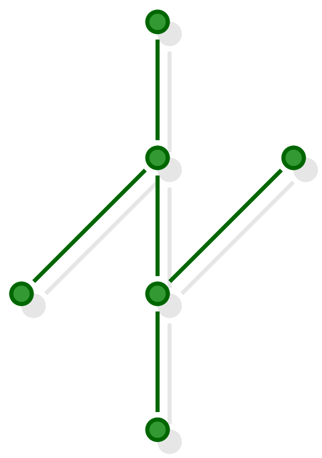
a_3 = maximal size of 3 antichains = 5

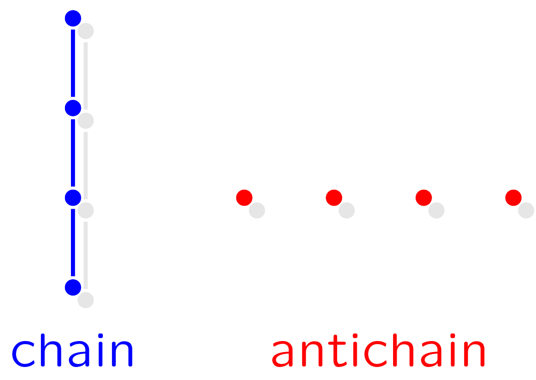
a_4 = maximal size of 4 antichains = 6

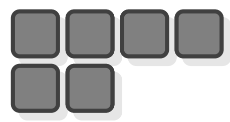


c_1 = maximal size of 1 chain = 4
c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2
a_2 = maximal size of 2 antichains = 4
a_3 = maximal size of 3 antichains = 5
a_4 = maximal size of 4 antichains = 6

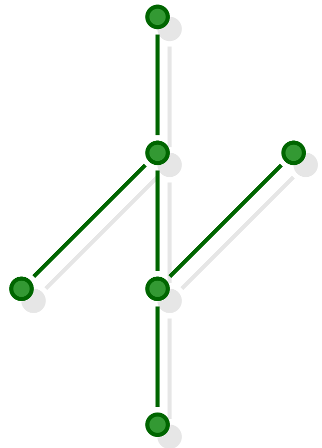


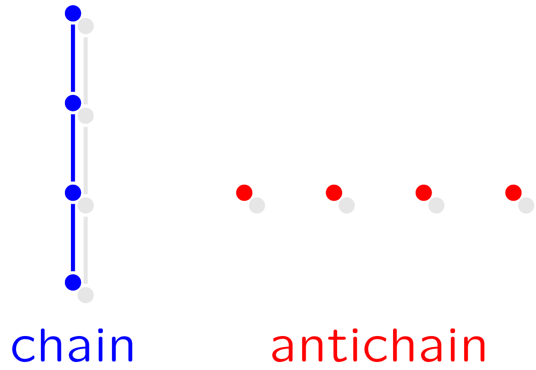




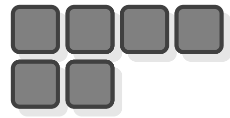
c_1 = maximal size of 1 chain = 4
c_2 = maximal size of 2 chains = 6

a_1 = maximal size of 1 antichain = 2
a_2 = maximal size of 2 antichains = 4
a_3 = maximal size of 3 antichains = 5
a_4 = maximal size of 4 antichains = 6

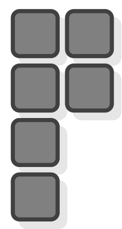


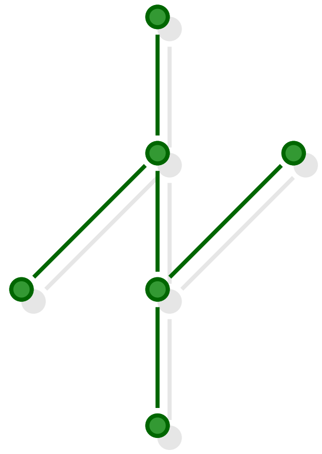


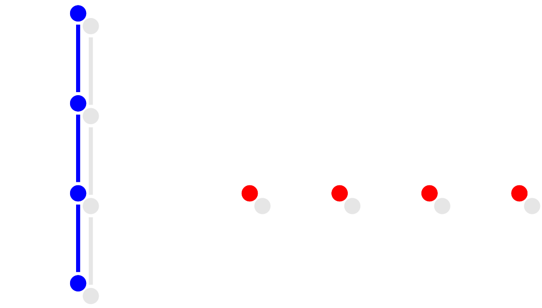
c_1 = maximal size of 1 chain = 4
c_2 = maximal size of 2 chains = 6



a_1 = maximal size of 1 antichain = 2
a_2 = maximal size of 2 antichains = 4
a_3 = maximal size of 3 antichains = 5
a_4 = maximal size of 4 antichains = 6







c_1 = maximal size of 1 chain = 4

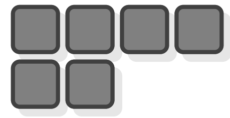
c_2 = maximal size of 2 chains = 6

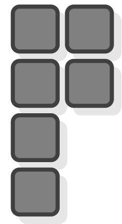
a_1 = maximal size of 1 antichain = 2

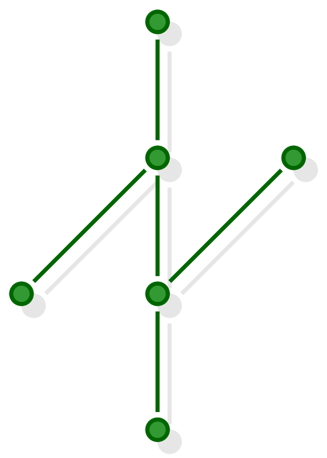
a_2 = maximal size of 2 antichains = 4

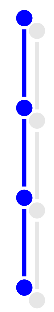
a_3 = maximal size of 3 antichains = 5

a_4 = maximal size of 4 antichains = 6

Δ = 

$\tilde{\Delta}$ = 

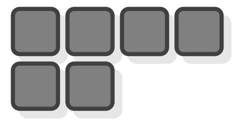




chain antichain

c_1 = maximal size of 1 chain = 4

c_2 = maximal size of 2 chains = 6

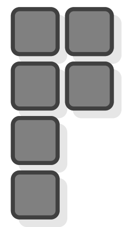
Δ = 

a_1 = maximal size of 1 antichain = 2

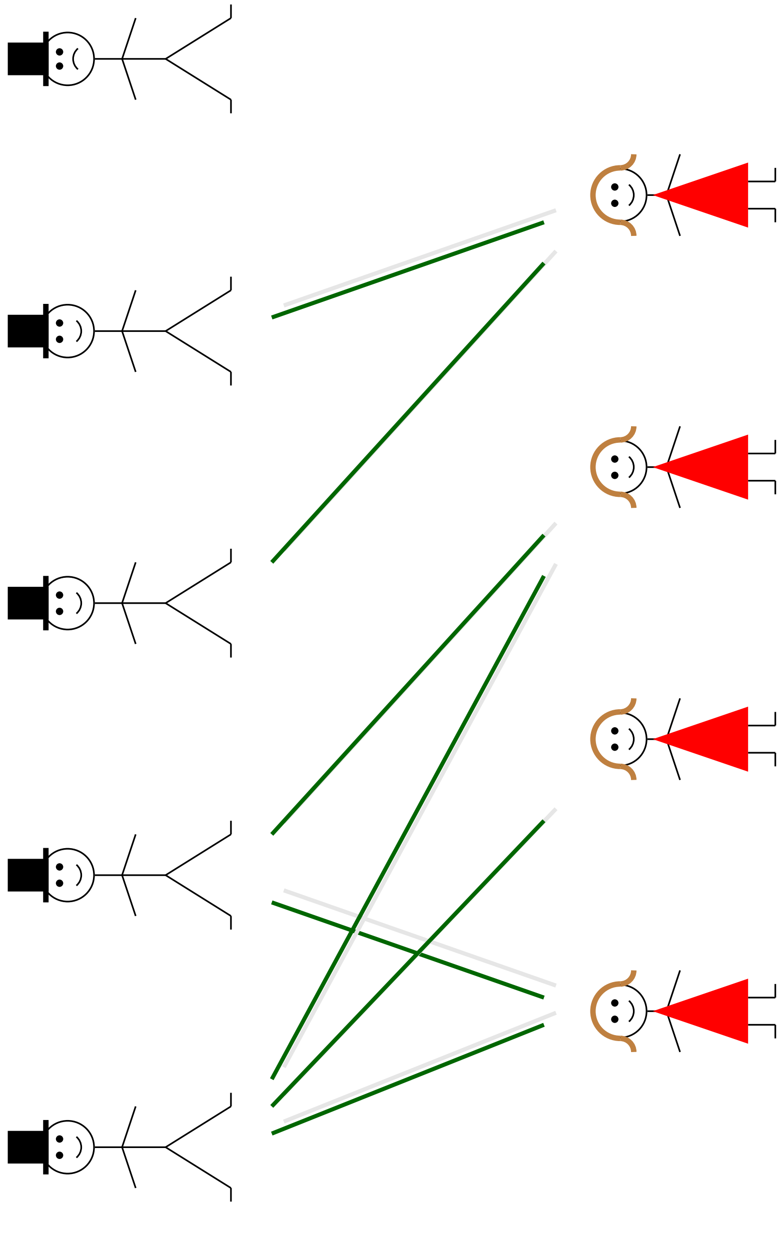
a_2 = maximal size of 2 antichains = 4

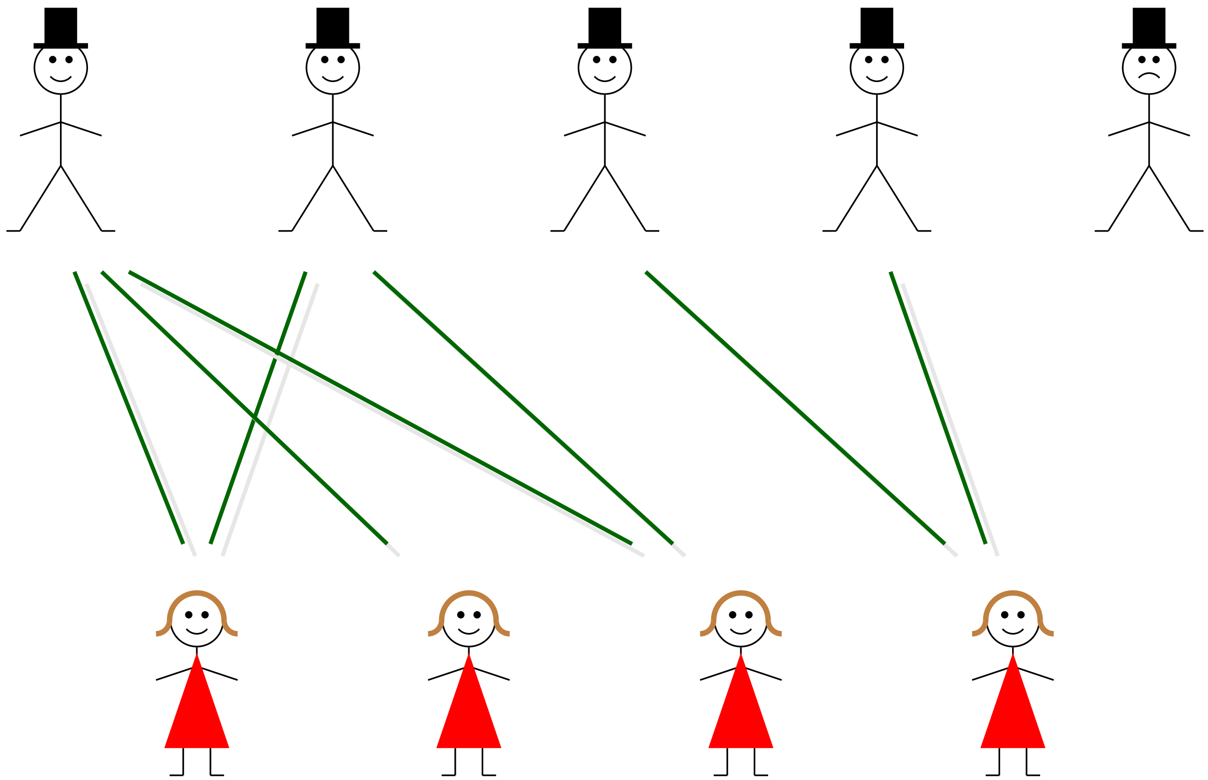
a_3 = maximal size of 3 antichains = 5

a_4 = maximal size of 4 antichains = 6

$\tilde{\Delta}$ = 

Greene's Theorem : $\Delta = \tilde{\Delta}^T$





The Marriage Theorem

The girls can all get a boy (each)

if and only if

k girls together like at least k boys $(\forall k \geq 1)$.

Duality Theorem 2:

The MacWilliams Identity

$\mathbb{F}$ = a field

$E = \{e_1, \ldots, e_n\}$

A subspace $C \subseteq \mathbb{F}^E$ is also called a *linear code*.

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Hamming distance: $d(u, v) = \#\{e \in E : u_e \neq v_e\}$

Hamming weight: $w(v) = d(v, 0)$

Support: $S(v) = \{e \in E : v_e \neq 0\}$

$\mathbb{F}$ = a field

$E = \{e_1, \ldots, e_n\}$

A subspace $C \subseteq \mathbb{F}^E$ is also called a *linear code*.

Hamming distance: $d(u, v) = \#\{e \in E : u_e \neq v_e\}$

Hamming weight: $w(v) = d(v, 0)$ $\qquad w(v) = 3$

Support: $S(v) = \{e \in E : v_e \neq 0\}$ $\qquad S(v) = \{e_2, e_3, e_5\}$

$$v = \begin{pmatrix} 0 & \overset{e_2}{\mathbf{1}} & \overset{e_3}{\mathbf{2}} & 0 & \overset{e_5}{\mathbf{-1}} \end{pmatrix}$$

$C \subseteq \mathbb{F}_q^E$ = a linear code with dimension k.

$$A_i = \#\{v \in C \; : \; w(v) = i\}$$

The weight enumerator $W_C(x,y)$ of C: $\quad W_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$

$C \subseteq \mathbb{F}_q^E$ = a linear code with dimension k.

$$A_i = \#\{v \in C : w(v) = i\}$$

The weight enumerator $W_C(x, y)$ of C: $\quad W_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} (0 & 0 & 0 & 0 & 0) \\ (1 & 0 & 1 & 0 & 0) \\ (0 & 1 & 1 & 0 & 0) \\ (0 & 0 & 0 & 1 & 1) \\ (1 & 1 & 0 & 0 & 0) \\ (1 & 0 & 1 & 1 & 1) \\ (0 & 1 & 1 & 1 & 1) \\ (1 & 1 & 0 & 1 & 1) \end{matrix} \qquad W_C(x,y) = x^5 + 4x^3y^2 + 3xy^4$$

$C \subseteq \mathbb{F}_q^E$ = a linear code with dimension k.

$$A_i = \#\{v \in C \,:\, w(v) = i\}$$

The weight enumerator $W_C(x,y)$ of C: $\quad W_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$

The MacWilliams Identity

$$W_{C^\perp}(x,y) = \frac{1}{q^k} W_C(x + (q-1)y, x - y)$$

$C \subseteq \mathbb{F}_q^E$ = a linear code with dimension k.

$$A_i = \#\{v \in C : w(v) = i\}$$

The weight enumerator $W_C(x,y)$ of C: $\quad W_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$

The MacWilliams Identity

$$W_{C^\perp}(x,y) = \frac{1}{q^k} W_C(x + (q-1)y, x - y)$$

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$M = (E,\rho)$ is a matroid if and only if, for all $X \subseteq Y \subseteq E$,

- $0 \leq \rho(X) \leq \rho(Y) \leq |Y|$
- $\rho(X) \cap \rho(Y) + \rho(X) \cup \rho(Y) \leq \rho(X) + \rho(Y)$

$C \subseteq \mathbb{F}_q^E$ = a linear code

$A_i^{(r)}$ = # r-dimensional subcodes $D \subseteq C$ with $|\bigcup_{v \in D} S(v)| = i$

The *rth higher weight enumerator* of C: $\quad W_C^{(r)}(z) = \sum_{i=0}^{n} A_i^{(r)} z^i$

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$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad W_C^{(2)}(z) = 3z^5 + 3z^4 + z^3$$

$$\begin{bmatrix} 1&1&0&0&0 \\ 1&0&1&0&0 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0 \\ 1&1&1&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&0&1&0&0 \\ 1&1&1&1&1 \end{bmatrix} \quad \begin{bmatrix} 0&1&1&0&0 \\ 1&1&1&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0 \\ 0&1&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&0&1&0&0 \\ 0&0&1&1&1 \end{bmatrix} \quad \begin{bmatrix} 0&1&1&0&0 \\ 0&0&1&1&1 \end{bmatrix}$$

3 5 5 5 4 4 4

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3	5	5	5	4	4	4

Problem: Calculate these enumerators for interesting linear codes

The extremal doubly-even, self-dual binary codes:

$1 \times [24, 12, 8]$ code (the extended binary Golay code)

$5 \times [32, 16, 8]$ codes

$1 \times [48, 24, 12]$ code

At most $1 \times [72, 36, 16]$ code

At least $12579 \times [40, 20, 8]$ codes

etc.

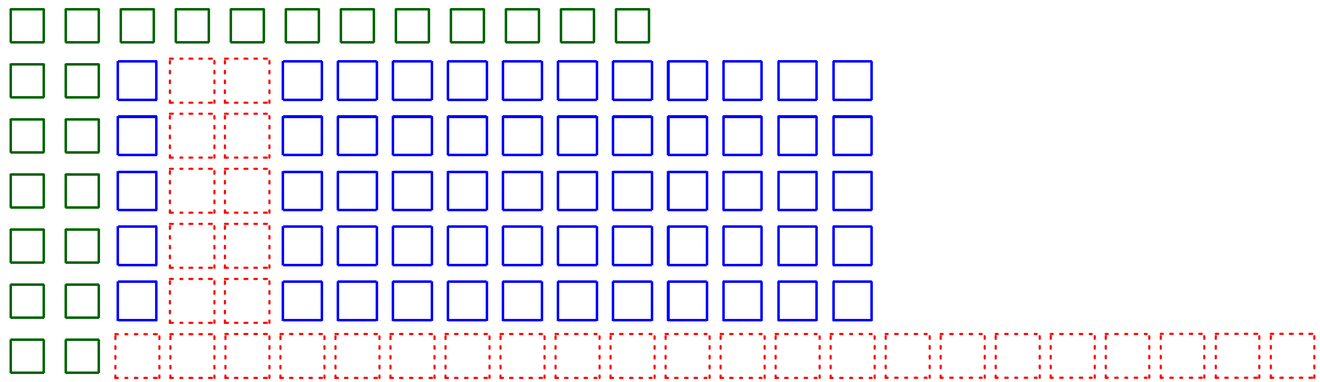
The extremal doubly-even, self-dual binary codes:

$1 \times [24, 12, 8]$ code

$5 \times [32, 16, 8]$ codes

$1 \times [48, 24, 12]$ code

$W_C^{(r)}(z)$:



□ : [Dougherty and Gulliver 2001]

□ : [Milenkovic, Coffey and Compton 2003]

□ : missing

The *Tutte polynomial* of a matroid $M = (E, \rho)$:

$$T_M(x, y) = \sum_{X \subseteq E} (x-1)^{\rho_M(E)-\rho_M(X)} (y-1)^{|X|-\rho_M(X)}$$

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Britz 2005: For a linear code $C \subseteq \mathbb{F}_q^E$ of dimension k,

$$W_C^{(r)}(z) = z^{n-k}(1-z)^k \sum_{i=0}^{r} \frac{(-1)^{r-i}}{[r]_r} q^{\binom{r-i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} T_{M_C}\left(\frac{1+(q^i-1)z}{1-z}, \frac{1}{z}\right)$$

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$$T_M(x, y) = \begin{cases} y\, T_{M/e}(x, y), & \rho_M(e) = 0; \\ x\, T_{M/e}(x, y), & \rho_{M^*}(e) = 0; \\ T_{M \setminus e}(x, y) + T_{M/e}(x, y), & \text{otherwise.} \end{cases}$$

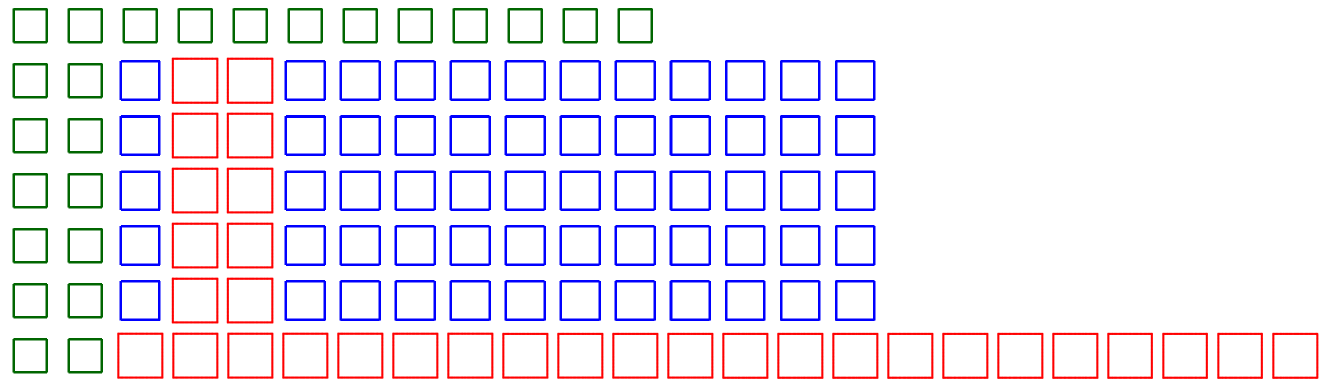
The extremal doubly-even, self-dual binary codes:

$1 \times [24, 12, 8]$ code

$5 \times [32, 16, 8]$ codes

$1 \times [48, 24, 12]$ code

$W_C^{(r)}(z)$:



□ : [Dougherty and Gulliver 2001]

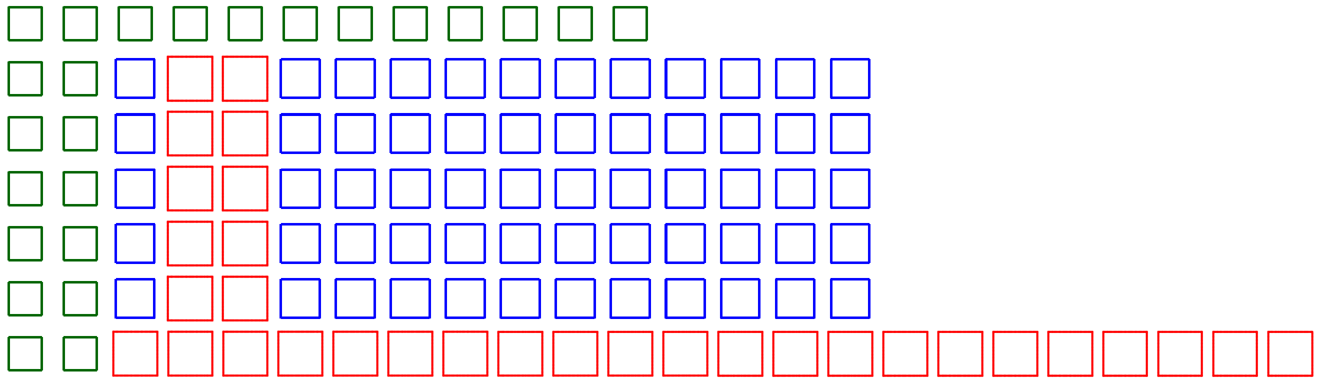
□ : [Milenkovic, Coffey and Compton 2003]

□ : [Britz, Britz, Shiromoto, and Sørensen 2007]

The extremal doubly-even, self-dual binary codes:

1 sec	$1 \times [24, 12, 8]$ code
40 sec	$5 \times [32, 16, 8]$ codes
800 hours	$1 \times [48, 24, 12]$ code

$W_C^{(r)}(z)$:



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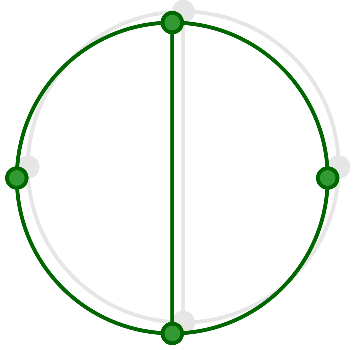
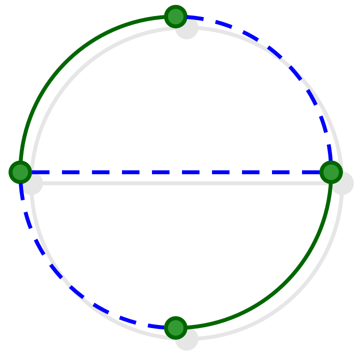
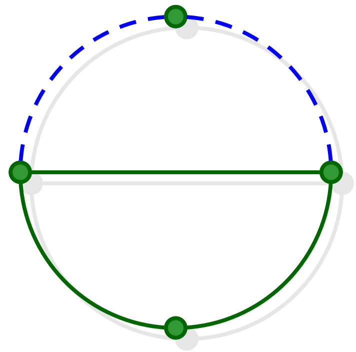
Duality Theorem 3:

Wei's Duality Theorem

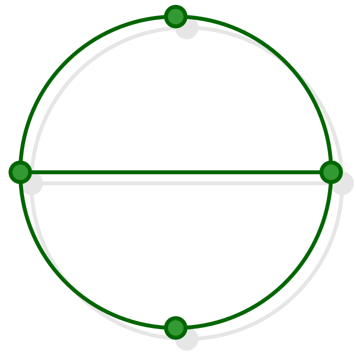




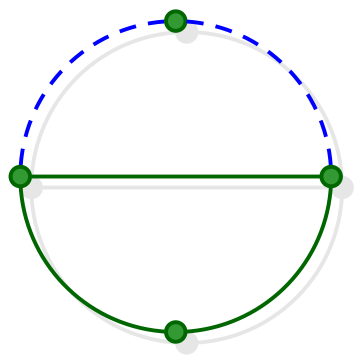
Bond = minimal cutset



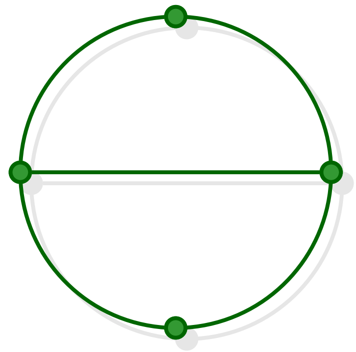
Bond = minimal cutset



b_1 = minimal size of a bond

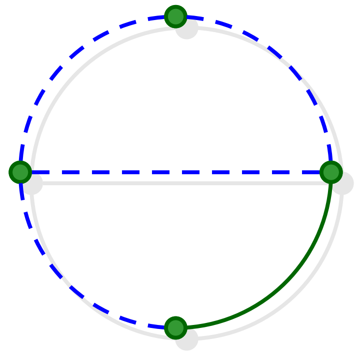


b_1 = minimal size of a bond = 2



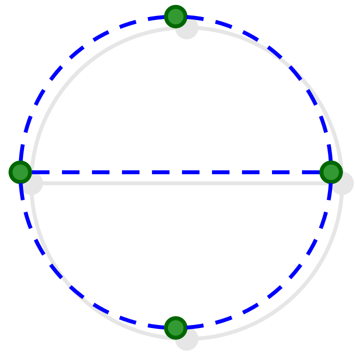
b_1 = minimal size of a bond = 2

b_2 = min. # edges in 2 distinct bonds =



b_1 = minimal size of a bond = 2

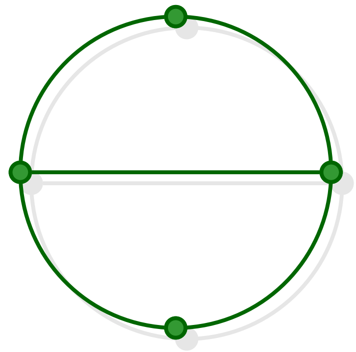
b_2 = min. # edges in 2 distinct bonds = 4



b_1 = minimal size of a bond = 2

b_2 = min. # edges in 2 distinct bonds = 4

b_3 = min. # edges in 3 distinct bonds $B_1, B_2, B_3,\ \ B_3 \not\subseteq B_1 \cup B_2$ = 5



b_1 = minimal size of a bond = 2

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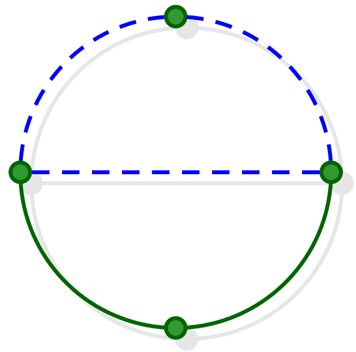
c_1 = minimal size of a cycle =

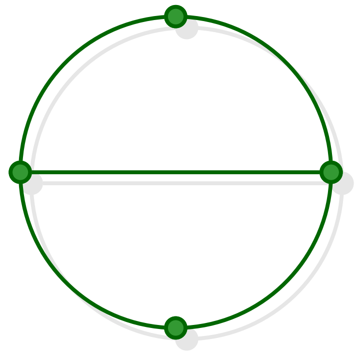


$b_1 =$ minimal size of a bond $= 2$

$b_2 =$ min. # edges in 2 distinct bonds $= 4$

$b_3 =$ min. # edges in 3 distinct bonds $B_1, B_2, B_3,\ \ B_3 \not\subseteq B_1 \cup B_2\ \ = 5$

$c_1 =$ minimal size of a cycle $= 3$



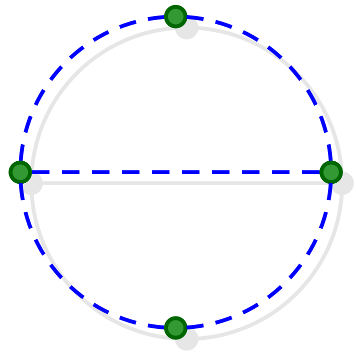
b_1 = minimal size of a bond = 2

b_2 = min. # edges in 2 distinct bonds = 4

b_3 = min. # edges in 3 distinct bonds $B_1, B_2, B_3,\ \ B_3 \not\subseteq B_1 \cup B_2$ = 5

c_1 = minimal size of a cycle = 3

c_2 = min. # edges in 2 distinct cycles =



b_1 = minimal size of a bond = 2

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c_1 = minimal size of a cycle = 3

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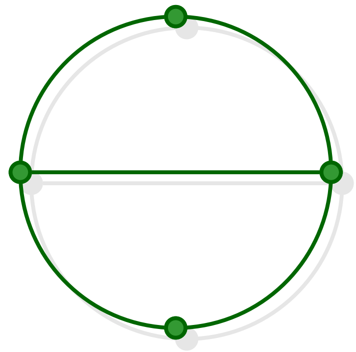


b_1 = minimal size of a bond = 2

b_2 = min. # edges in 2 distinct bonds = 4

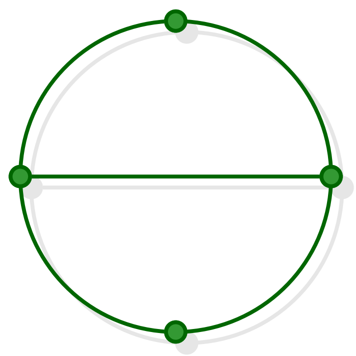
b_3 = min. # edges in 3 distinct bonds B_1, B_2, B_3, $B_3 \not\subseteq B_1 \cup B_2$ = 5

c_1 = minimal size of a cycle = 3

c_2 = min. # edges in 2 distinct cycles = 5

Set $U = \{b_1, b_2, b_3\} = \{2, 4, 5\}$

and $V = \{5 + 1 - c_2, 5 + 1 - c_1\} = \{1, 3\}$.



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Set $U = \{b_1, b_2, b_3\} = \{2, 4, 5\}$

and $V = \{5 + 1 - c_2, 5 + 1 - c_1\} = \{1, 3\}$.

$U \cup V = \{1, 2, 3, 4, 5\}$ and $U \cap V = \emptyset$

G = a multigraph on n edges

Define

k = # edges in a spanning forest of G

b_i = min. # edges in i bonds, none contained in the union of the others

c_j = min. # edges in j cycles, none contained in the union of the others

$U = \{b_1, \ldots, b_k\}$

$V = \{n + 1 - c_{n-k}, \ldots, n + 1 - c_1\}$.

Britz 2007: $U \cup V = \{1, \ldots, n\}$ and $U \cap V = \emptyset$.

$M =$ a matroid of rank k on n elements

Define

$f_i =$ maximal size of an i-rank set in M

$f_j^* =$ maximal size of an j-rank set in M^*

$U = \{f_0 + 1, \ldots, f_{k-1} + 1\}$

$V = \{n - f^*_{n-k-1}, \ldots, n - f_0^*\}.$

Britz et al.: $U \cup V = \{1, \ldots, n\}$ and $U \cap V = \emptyset$.

M = a matroid of rank k on n elements

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Britz et al.: $U \cup V = \{1, \ldots, n\}$ and $U \cap V = \emptyset$.

Proof. Assume that the theorem is false.

Then $f_i + 1 = n - f_j^*$ for some i, j.

Let $A \subseteq E$ satisfy $|A| = f_j^*$ and $r_{M^*}(A) = j$.

Then $|E - A| = f_i + 1$, so $r_M(E - A) \geq i + 1$.

Since $|E - A| + r_{M^*}(A) - r(M^*) = r_M(E - A)$,

$$-f_j^* + j + r \geq i + 1.$$

Similarly,

$$n - f_i + i - r \geq j + 1.$$

Hence, $1 = n - f_i - f_j^* \geq 2$, a contradiction. □

M = a matroid of rank k on n elements

Define

f_i = maximal size of an i-rank set in M

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Britz et al.: $U \cup V = \{1, \ldots, n\}$ and $U \cap V = \emptyset$.

Britz et al.: Further generalizations.

Britz et al.: Applications to graphs, codes, modules, and matchings.

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Wei's Duality Theorem

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Wei's Duality Theorem

Thank you!