THE GENERALIZED INVERSE $A^{(2)}_{T,S}$ OF A MATRIX OVER AN ASSOCIATIVE RING

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Abstract

In this paper we establish the definition of the generalized inverse $A^{(2)}_{T,S}$, which is a $[2]$ inverse of a matrix $A$ with prescribed image $T$ and kernel $S$ over an associative ring, and give necessary and sufficient conditions for the existence of the generalized inverse $A^{(2)}_{T,S}$ and some explicit expressions for $A^{(1,2)}_{T,S}$ of a matrix $A$ over an associative ring, which reduce to the group inverse or $[1]$ inverses. In addition, we show that for an arbitrary matrix $A$ over an associative ring, the Drazin inverse $A_d$, the group inverse $A_g$, and the Moore-Penrose inverse $A^\dagger$, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$.


1. Introduction

It is well known that, over the field of complex numbers, the Moore-Penrose inverse, the Drazin inverse, the group inverse and so on, are all the generalized inverse $A^{(2)}_{T,S}$, which is a $[2]$ inverse of a matrix $A$ with prescribed range $T$ and null space $S$ (see [2, 10]). Y. Wei in [11] gave an explicit expression for the generalized inverse $A^{(2)}_{T,S}$ which reduces to the group inverse.

There are some results on generalized inverses of matrices, such as the Drazin inverse, the group inverse and the Moore-Penrose inverse, over an associative ring (see, for example, [3]–[8]). These results include necessary and sufficient conditions for the existence of these generalized inverses. In [8], Corollary 1 implies that over an associative ring, a von Neumann regular matrix $A$ has a group inverse if and only if $A^2A^{(1)} + I - AA^{(1)}$ is invertible, if and only if $A^{(1)}A^2 + I - A^{(1)}A$ is invertible. Recently,
similar results about the Moore-Penrose inverse and the Drazin inverse appeared in [6, 7]. This is a motivation for our research.

Throughout this paper, \( R \) denotes an associative ring with identity 1 and \( R^{m \times n} \) denotes the set of \( m \times n \) matrices over \( R \). In particular, we write \( R^m \) for \( R^{m \times 1} \) and \( M_n(R) \) for \( R^{n \times n} \), the ring of square \( n \times n \) matrices over \( R \). By a module we mean a right \( R \)-module. If \( S \) is an \( R \)-submodule of an \( R \)-module \( M \) then we write \( S \subseteq M \).

Let \( A \in R^{m \times n} \). We denote the image of \( A \) (that is \( \{Ax | x \in R^n \} \)) by \( R(A) \) and the kernel of \( A \) (that is \( \{x \in R^n | Ax = 0 \} \)) by \( N(A) \).

An \( m \times n \) matrix \( A \) over \( R \) is said to be \textit{von Neumann regular} if there exists an \( n \times m \) matrix \( X \) over \( R \) such that

\[
(1) \quad AXA = A.
\]

In this case \( X \) is called a \([1]\) \textit{inverse} of \( A \) and is denoted by \( A^{(1)} \).

An \( n \times n \) matrix \( A \) over \( R \) is said to be \textit{Drazin invertible} if for some positive integer \( k \) there exists a matrix \( X \) over \( R \) such that

\[
(2) \quad A^kXA = A^k, \\
(3) \quad XAX = X, \\
(4) \quad AX =XA.
\]

If \( X \) exists then it is unique and is called the \textit{Drazin inverse} of \( A \) and denoted by \( A_d \). If \( k \) is the smallest positive integer such that \( X \) and \( A \) satisfy (2), (3) and (4), then it is called the \textit{Drazin index} and denoted by \( k=\text{Ind}(A) \). If \( k = 1 \) then \( A_d \) is denoted by \( A_s \) and is called the \textit{group inverse} of \( A \).

Let \( * \) be an involution on the matrices over \( R \). Recall that an \( m \times n \) matrix \( A \) over \( R \) is said to be \textit{Moore-Penrose invertible} (with respect to \( * \)) if there exists an \( n \times m \) matrix \( X \) such that (1) and (3) hold and

\[
(6) \quad (AX)^* = AX, \\
(7) \quad (XA)^* =XA.
\]

If \( X \) exists then it is unique and is called the \textit{Moore-Penrose inverse} of \( A \) and denoted by \( A^\dagger \). If a matrix \( X \) satisfies condition (3) then \( X \) is called a \([2]\) \textit{inverse} of \( A \).

In Section 2 we shall establish the definition of the generalized inverse \( A^{(2)}_{T,S} \), which is a \([2]\) inverse of a matrix \( A \) over an associative ring with prescribed image \( T \) and kernel \( S \), and show that for an arbitrary matrix \( A \) over an associative ring the Drazin inverse \( A_d \), the group inverse \( A_s \) and the Moore-Penrose inverse \( A^\dagger \), if they exist, are all the generalized inverse \( A^{(2)}_{T,S} \). In Section 3, we give necessary and sufficient conditions for the existence of the generalized inverse \( A^{(2)}_{T,S} \). In Section 4 we study some explicit expressions for \( A^{(1,2)}_{T,S} \) of a matrix \( A \) over an associative ring, which reduce to the group inverse or \([1]\) inverses, and some equivalent conditions for the existence of \( A^{(1,2)}_{T,S} \).
2. The generalized inverse $A_{T,S}^{(2)}$

Suppose that $L, M \subset R^n$ and $L \oplus M = R^n$. Then every $x \in R^n$ can be uniquely written as $x = x_1 + x_2$, where $x_1 \in L, x_2 \in M$. Thus

$$P_{L,M}x = x_1$$

defines a homomorphism $P_{L,M} : R^n \rightarrow R^n$ called the projection of $R^n$ on $L$ along $M$. This homomorphism can be represented by a matrix with respect to the standard basis of $R^n$, since the module $R^n$ is free. The symbol $P_{L,M}$ is used to denote the matrix as well.

About $P_{L,M}$, we have the following results, whose proof is analogous to that over the field of complex numbers.

**LEMMA 2.1.** If $L, M \subset R^n$ and $L \oplus M = R^n$ then

(i) $P_{L,M}A = A$ if and only if $R(A) \subset L$,

(ii) $AP_{L,M} = A$ if and only if $N(A) \supset M$.

We now characterize the $(2)$ inverse of a matrix $A$ over $R$ with prescribed image $T$ and kernel $S$. The proof of the following theorem is analogous to that of [13, Theorem 1].

**THEOREM 2.2.** Let $A$ be an $m \times n$ matrix over an associative ring $R$ with identity and $T \subset R^n$ and $S \subset R^m$. Then the following conditions are equivalent.

(i) There exists some $X \in R^{n \times m}$ such that

$$XAX = X, \quad R(X) = T, \quad N(X) = S.$$  \hspace{2cm} (2.1)

(ii) $AT \oplus S = R^m$ and $N(A) \cap T = \{0\}$.

If these conditions are satisfied then $X$ is unique.

**PROOF.** (i)$\Rightarrow$(ii) Since $XAX = X$, $AX$ is an idempotent homomorphism from $R^m$ to $R^m$. So, by [1, Lemma 5.6],

$$R(AX) \oplus N(AX) = R^m.$$  

It is easy to see that $R(AX) = AR(X) = AT$ and $N(AX) = N(X) = S$. Hence

$$AT \oplus S = R^m.$$  

Next we will show that $N(A) \cap T = \{0\}$. Let $x \in N(A) \cap T$. Then $Ax = 0$ and there exists a $y \in R^n$ such that $x = Xy$. So $x = Xy = XAXy = XAx = 0$. Therefore we have $N(A) \cap T = \{0\}$. 


(ii)⇒(i) Obviously $A|_T$ is an epimorphism from $T$ to $AT$. Since $N(A|_T) = N(A) \cap T = 0$, $A|_T$ is a monomorphism and so $A|_T$ has an inverse $(A|_T)^{-1} : AT \to T$.

From $AT \oplus S = R^m$, we know that any $y \in R^m$, can be uniquely written as $y = y_1 + y_2$, where $y_1 \in AT, y_2 \in S$. So we define $X : R^m \to R^n$ by $Xy = (A|_T)^{-1}y_1$.

Obviously $X$ is a homomorphism and satisfies

\[
\begin{align*}
Xy &= (A|_T)^{-1}y, & \text{if } y \in AT; \\
Xy &= 0, & \text{if } y \in S.
\end{align*}
\]

Because $R^m$ and $R^n$ are both free modules, there exists a matrix of the homomorphism $X$ with respect to the standard bases of $R^m$ and $R^n$, and we write $X$ for the matrix as well. It is easy to see that $R(X) = T$ and $N(X) = S$ by $AT \oplus S = R^m$.

For every $y \in R^m = AT \oplus S$ we have $y = y_1 + y_2$ where $y_1 \in AT, y_2 \in S$. Then

\[
XAXy = XAXy_1 = X(A|_T)^{-1}y_1 = Xy_1 = Xy.
\]

This implies that $XAX = X$.

Now we prove the uniqueness. Suppose that $X_1$ and $X_2$ both satisfy (2.1). Then $X_1A$ and $AX_2$ are idempotent matrices of order $m$ and $n$ respectively, and

\[
X_1A = P_{R(X_1), N(X_1)} = P_{R(X_1), N(X_1)} = P_{R, N(X_1)} = P_{R, N(X_1)} = P_{R, N(X_1)},
\]

\[
AX_2 = P_{R(X_2), N(X_2)} = P_{R(X_2), N(X_2)} = P_{R(X_2), N(X_2)} = P_{R(X_2), S}.
\]

By Lemma 2.1, we deduce that

\[
X_2 = P_{R, N(X_1)}X_2 = (X_1A)X_2 = X_1AX_2 = X_1P_{R(X_2), S} = X_1 = X_2
\]

A matrix $X \in R^{n \times m}$ is called the generalized inverse which is a \{2\} inverse of a matrix $A$ over $R$ with prescribed image $T$ and kernel $S$ if it satisfies the equivalent conditions in Theorem 2.2, and is denoted by $A_{T,S}^{(2)}$.

By (2.2), we have that

\[
A_{T,S}^{(2)} = (A|_T)^{-1}P_{AT,S}.
\]

From the proof of uniqueness in the theorem above and Lemma 2.1, we have the following corollary.

**Corollary 2.3.** Let $A$ and $G$ be matrices over an associative ring $R$. If the generalized inverse $A_{T,S}^{(2)}$ exists, then

(i) $A_{T,S}^{(2)}AG = G$ if and only if $R(G) \subseteq T$;

(ii) $GAA_{T,S}^{(2)} = G$ if and only if $N(G) \supset S$.

About the generalized inverse, we also have the following property.
**Theorem 2.4.** Let $A$ be a matrix over $R$. If $A^{(2)}_{T,S}$ exists and there exists a matrix $G$ over $R$ satisfying $R(G) = T$ and $N(G) = S$ then there exists a matrix $W$ over $R$ such that

\[(2.4) \quad GAGW = G, \]
\[(2.5) \quad A^{(2)}_{T,S}AGW = A^{(2)}_{T,S}. \]

**Proof.** Suppose $A^{(2)}_{T,S}$ exists with $R(G) = T$ and $N(G) = S$ for a matrix $G$. Then $AR(G) \oplus N(G) = R^m$ and so there exists an epimorphism $R^m \rightarrow N(G) \rightarrow 0$. By [1, Theorem 8.1], $N(G)$ has a finite spanning set whose elements constitute a matrix, denoted by $L$. Thus $GL = 0$, and the columns of $(AG, L)$ generate $R^m$, that is, there exists a matrix $(W^T, W_1^T)$ such that

\[
AGW + LW_1 = I_m.)
\]

If we multiply the left hand side by $G$ and $A^{(2)}_{T,S}$ respectively, then we obtain (2.4) and (2.5).

The following theorem shows that for an arbitrary matrix $A$ over an associative ring, $A^\dagger$, $A_d$ and $A_g$, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$.

**Theorem 2.5.**

(i) Let $A$ be an $m \times n$ matrix over $R$ and let $*$ be an involution on the matrices over $R$. If $A^\dagger$ exists, then $A^\dagger = A^{(2)}_{R(A^\dagger), N(A^\dagger)}$.

(ii) Let $A$ be an $n \times n$ matrix over $R$, and $k = \text{Ind}(A)$. If $A_d$ exists, then $A_d = A^{(2)}_{R(A_d), N(A_d)}$.

(iii) Let $A$ be an $n \times n$ matrix over $R$. If $A_g$ exists, then $A_g = A^{(2)}_{R(A), N(A)}$.

**Proof.**

(i) Since $A^\dagger \in A[1, 2]$ and $A^\dagger \in A^\star[1, 2]$, we easily see that

\[
R(A^\dagger) = R((A^\dagger)^\dagger) = R(A^\dagger A^\dagger) = R(A^\dagger),
\]
\[
N(A^\dagger) = N(AA^\dagger) = N((AA^\dagger)^\dagger) = N(A^\dagger A^\dagger) = N(A^\dagger),
\]

and $N(A) = N(A^\dagger A)$.

Since $AA^\dagger$ and $A^\dagger A$ are idempotent, we have

\[
R^m = R(AA^\dagger) \oplus N(AA^\dagger) = AR(A^\dagger) \oplus N(AA^\dagger) = AR(A^\dagger) \oplus N(A^\dagger)
\]

and

\[
N(A) \cap R(A^\dagger) = N(A^\dagger A) \cap R(A^\dagger A) = \{0\}
\]

by [1, Lemma 5.6]. So, by Theorem 2.2, $A^{(2)}_{R(A^\dagger), N(A^\dagger)}$ exists and $A^\dagger = A^{(2)}_{R(A^\dagger), N(A^\dagger)}$. 

(ii) Firstly, we shall show that
\[ R(A_d) = R(AA_d) = R(A^l) \quad \text{and} \quad N(A_d) = N(AA_d) = N(A^l) \]
for any positive integer \( l \geq k \). Since
\[ R(A_d) = R(AA_d^2) \subseteq R(AA_d) = R(A_dA) \subseteq R(A_d), \]
we have \( R(A_d) = R(AA_d) \) and so
\[ R(AA_d) = AR(A_d) = AR(AA_d) = A^2R(A_d). \]

It is easy to obtain inductively that \( R(AA_d) = A^h R(A_d) \) for any positive integer \( h \).
This gives us that \( R(A_d) = R(AA_d) = R(A^l) \) for any positive integer \( l \geq k \). Also, since for any positive integer \( l \geq k \),
\[ N(A_d) \subset N(A^{l+1}A_d) = N(A^l) \subset N(A_dA^l) = N(A_dA) \subset N(A_d^2A) = N(A_d), \]
we get that \( N(A_d) = N(AA_d) = N(A^l). \)

Since \( AA_d \) is idempotent, by [1, Lemma 5.6], we have
\[ R^n = R(AA_d) \oplus N(AA_d) = AR(A^l) \oplus N(A^l) = R^n. \]

Since
\[ N(A) \cap R(A^l) \subset N(A^l) \cap R(A^{l+1}) = \{0\}, \]
\( A_{R(A^l), N(A^l)}^{(2)} \) exists and \( A_{R(A^l), N(A^l)}^{(2)} = A_d \) by Theorem 2.2.

(iii) Take \( k = 1 \) in (ii).

\[ \square \]

3. The generalized inverse \( A_{T,S}^{(1,2)} \)

If the generalized inverse \( A_{T,S}^{(2)} \) satisfies \( AA_{T,S}^{(2)} = A \) then it is called the generalized inverse which is a \( \{1,2\} \) inverse of a matrix \( A \) over \( R \) with prescribed image \( T \) and kernel \( S \), and is denoted by \( A_{T,S}^{(1,2)} \). (Its uniqueness is guaranteed by the following theorem.)

**Theorem 3.1.** Let \( A \) be an \( m \times n \) matrix over an associative ring \( R \) with identity and \( T \subset R^n \) and \( S \subset R^m \). Then the following conditions are equivalent.

(i) \( AT \oplus S = R^n, \quad R(A) \cap S = \{0\} \) and \( N(A) \cap T = \{0\} \).
(ii) \( R(A) \oplus S = R^n, \quad N(A) \oplus T = R^n \).
(iii) There exists some \( X \in R^{m \times n} \) such that
\[ AXA = A, \quad XAX = X, \quad R(X) = T, \quad N(X) = S. \]

If these conditions are satisfied then \( X \) is unique.
The generalized inverse $A_{ij}^{(0)}$

**Proof.** (ii) $\Rightarrow$ (i) It is obvious that $R(A) \cap S = \{0\}$ and $N(A) \cap T = \{0\}$. To obtain $AT \oplus S = R^m$, it suffices to prove $AT = R(A)$.

Obviously, $AT \subset R(A)$. For any $x \in R(A)$, we have $x = Ay$, where $y \in R^n$. Since $N(A) \oplus T = R^n$, we can write $y = y_1 + y_2$, where $y_1 \in N(A), y_2 \in T$. Thus,

$$x = Ay = Ay_1 + Ay_2 = Ay_2 \in AT,$$

and therefore $R(A) \subset AT$. Consequently, $AT = R(A)$.

(i) $\Rightarrow$ (iii) By Theorem 2.2, from $AT \oplus S = R^m$ and $N(A) \cap T = \{0\}$, we know that $X = A_{ij}^{(0)}$ exists and that $R(X) = T$ and $N(X) = S$. We shall show $AXA = A$.

Since $XAX = X$, we have $XAXA = AX$ and then $X(AXA - A) = 0$. So

$$R(AXA - A) \subset R(A) \cap N(X) = R(A) \cap S = \{0\}.$$

Hence $AXA = A$.

(iii) $\Rightarrow$ (ii) From (iii), we have $(AX)^2 = AX$, $(XA)^2 =XA$, and

$$N(X) \subset N(AX) \subset N(XAX) = N(X),$$

$$N(XA) \subset N(AXA) = N(A) \subset N(XA),$$

$$R(XA) \subset R(X) = R(XAX) \subset R(XA),$$

$$R(AX) \subset R(A) = R(AXA) \subset R(AX).$$

So

$$N(AX) = N(X) = S, \quad N(XA) = N(A),$$

$$R(XA) = R(X) = T, \quad R(AX) = R(A).$$

By [1, Lemma 5.6] and the four equations above, we reach (ii).

By Theorem 2.2, $X$ is unique.

The next result is concerning the equivalent conditions in Theorem 3.1.

**Theorem 3.2.** Let $A$ be an $m \times n$ matrix over an associative ring $R$ with identity $T \subset R^n$ and $S \subset R^m$.

(i) If $N(A) + T = R^n$ then $AT = R(A)$.

(ii) If $AT \oplus S = R^m$ then

$$AT = R(A) \quad \text{if and only if} \quad R(A) \cap S = \{0\}.$$

**Proof.** (i) From the proof of the theorem above (ii) implies (i).

(ii) Suppose that $R(A) \cap S = \{0\}$. Obviously, $AT \subset R(A)$. Now we will show the inclusion in reverse. For any $x \in R(A)$,

$$x = x_1 + x_2 \in R^m = AT \oplus S,$$
where \( x_1 \in AT, x_2 \in S \). By \( AT \subset R(A), x_1 \in R(A) \). So
\[
x_2 = x - x_1 \in R(A) \cap S = \{0\}.
\]
Therefore, \( x_2 = 0 \) and then \( x = x_1 \in AT \). Hence \( R(A) \subset AT \).

Conversely, suppose that \( AT = R(A) \). Since \( AT \oplus S = R^n \) and \( AT = R(A) \), we have \( R(A) \cap S = AT \cap S = \{0\} \).

We denote the maximal order of a nonvanishing minor of \( A \) over a commutative ring \( R \) by \( \rho(A) \). This is called the determinantal rank of \( A \). Obviously \( \rho(AB) \leq \min\{\rho(A), \rho(B)\} \) (see [9, Theorem 2.3]). When \( R \) is the complex number field, \( \rho(A) = \text{rank}(A) \).

**THEOREM 3.3.** Let \( A \) be an \( m \times n \) matrix over an integral domain \( R \) and \( T \subset R^n \) and \( S \subset R^n \) be free submodules. If \( AT \oplus S = R^n \) then the following conditions are equivalent.

(i) \( N(A) \cap T = \{0\} \) and \( R(A) \cap S = \{0\} \).

(ii) \( \dim(T) = \rho(A) \) and \( \dim(S) = m - \dim(T) \).

**PROOF.** Suppose that (i) holds and let the columns of \( U \) be a basis of \( T \). From the proof of [13, Theorem 2], we have \( \dim(T) = \dim(AT) = \rho(AU) \leq \rho(A) \) and \( \dim(S) = m - \dim(T) \). By Theorem 3.2, \( AT = R(A) \). Thus there exists a matrix \( X \) over \( R \) such that \( A = AU \). Thus \( \rho(A) \leq \rho(AU) = \dim(AT) \). Therefore \( \rho(A) = \dim AT = \dim(T) \).

Conversely, suppose that (ii) holds. We have that \( \dim(T) = \dim(AT) \) from the proof of [13, Theorem 2]. Thus \( \rho(A) = \dim(T) = \dim(AT) \). By [12, Lemma 1], the maximal number of linearly independent columns of \( A \) is \( \dim(AT) \). Since \( AT \subset R(A), R(A) + S = R^n \). Over the quotient field \( F \) of \( R \), \( AT = R(A) \) because \( \rho(A) = \dim(AT) \), and \( R(A) \oplus S = R^n \). Therefore \( x \) and \( y \) are linear independent over \( F \) for any \( x \in R(A), y \in S \).

On the other hand, over an integral domain \( R \), suppose that \( 0 \neq z \in R(A) \cap S \). Then there exist \( r_i \in R, i = 1, \ldots, s \), such that

\[
z = \sum_{i=1}^{s} \beta_i r_i,
\]
where \( \{\beta_1, \beta_2, \ldots, \beta_s\} \) is a basis of \( S \) and \( s = \dim(S) \). But Equation (3.1) is true over \( F \). This is in contradiction to the above reasoning. Hence \( R(A) \cap S = \{0\} \).

The remainder of the proof is obtained from [13, Theorem 2].

**REMARK 1.** A module over the field of complex numbers is a vector space. So when \( R \) is the field of complex numbers, the above theorem ensures that Theorem 3.1 extends [2, Corollary 2.10].
4. Explicit expressions for $A_{T,S}^{(1,2)}$

We now consider some explicit expressions for $A_{T,S}^{(1,2)}$, which reduce to the group inverse or $\{1\}$ inverses. Firstly we shall prove the following lemma. In the proof, we use the following fact.

**Proposition 4.1.** If $e$ is idempotent in a ring $R$ with identity 1 and $x, y \in eRe$ then $xy = e$ if and only if $(x + 1 - e)(y + 1 - e) = 1$.

**Lemma 4.2.** Let $A$ be an $m \times n$ von Neumann regular matrix over $R$ and $G$ an $n \times m$ matrix over $R$. Then $U = AGAA^{(1)} + I_m - AA^{(1)}$ is invertible if and only if $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible.

**Proof.** If $U$ is invertible then there exists an $X$ such that $UX = XU = I_m$. That is,

$$(AGAA^{(1)} + I_m - AA^{(1)})X = I_m \quad \text{and} \quad X(AGAA^{(1)} + I_m - AA^{(1)}) = I_m.$$ 

Multiplying on the left by $A^{(1)}AA^{(1)}$ and the right by $A$ and, since $A = AA^{(1)}A$, we have

$$(A^{(1)}AGA)(A^{(1)}AA^{(1)}XA) = A^{(1)}A \quad \text{and} \quad (A^{(1)}AA^{(1)}XA)(A^{(1)}AGA) = A^{(1)}A.$$ 

Since $A^{(1)}AGA = A^{(1)}A(GA)A^{(1)}A$ and $A^{(1)}AA^{(1)}XA = A^{(1)}A(A^{(1)}XA)A^{(1)}A$, we know that $A^{(1)}AGA$ has the inverse matrix $A^{(1)}AA^{(1)}XA$ in $A^{(1)}AM_n(R)A^{(1)}A$. Thus $V = A^{(1)}AGA + I_n - A^{(1)}A$ has the inverse matrix

$$A^{(1)}A \left( A^{(1)}AA^{(1)}XA \right) A^{(1)}A + I_n - A^{(1)}A \quad \text{in} \quad M_n(R).$$ 

The proof of the converse is analogous. □

Next we shall show the main result of this section. The following theorem not only shows some explicit expressions for $A_{T,S}^{(1,2)}$ which reduce to the group inverse or $\{1\}$ inverses, but also gives some equivalent conditions for the existence of $A_{T,S}^{(1,2)}$.

**Theorem 4.3.** Let $A$ be an $m \times n$ matrix over $R$ and $G$ an $n \times m$ matrix over $R$. Then the following conditions are equivalent.

(i) $A$ is von Neumann regular, $U = AGAA^{(1)} + I_m - AA^{(1)}$ is invertible and $N(A) \cap R(G) = \{0\}$.

(ii) $A$ is von Neumann regular, $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible and $N(A) \cap R(G) = \{0\}$.

(iii) $A_{R(G), N(G)}^{(1,2)}$ exists.
When these conditions are satisfied we have

\[ A^{(1,2)}_{R(G), N(G)} = G(AG)_{\#} = (GA)_{\#}G \]

(4.2) \[ = G(GAG)^{(I)} \]

(4.3) \[ = G(GA)^{(I)} A(GA)^{(I)} G \]

(4.4) \[ = G^{-2}AG = G^{-1}AV^{-1}G = GAV^{-2}G. \]

**Proof.** (i) and (ii) are equivalent by Lemma 4.2.

To show that (ii) implies (iii), set \( B = AV^{-2}G \). Using \( UA = AGA = AV \), we have \( B = (AG)_{\#} \) because

\[ B(AG) = AV^{-2}GAG = U^{-2}AGAG = U^{-1}AG = AV^{-1}G = AGAV^{-2}G = (AG)B, \]

\[ B(AG)B = U^{-1}AG(AV^{-2}G) = AV^{-2}G = B, \]

(4.5) \[ (AG)B(AG) = (AG)AV^{-1}G = AG. \]

Analogously, we deduce that \((GA)_{\#}\) exists and \((GA)_{\#} = GU^{-2}A\). Let \( X = (GAG)_{\#} \).

It is obvious that

(4.5) \[ XAX = X. \]

Since

\[ AG = (AG)^{2}(AG)_{\#} = AGAX, \]

we have \( A(G - GAX) = 0 \) and then

\[ R(G - GAX) = R(G(I - AX)) \subset N(A) \cap R(G) = \{0\}. \]

Therefore

(4.6) \[ G = GAX = GA(GAG)_{\#} = G(GA)_{\#}AG \]

(4.7) \[ = XAG. \]

Using (4.6) and (4.7), we have

(4.8) \[ R(X) = R(G) \quad \text{and} \quad N(X) = N(G). \]

Since \( AV = AGA \), we get

\[ A = AGAV^{-1} = AG(AG)_{\#}AGAV^{-1} = AXA. \]

Using the equation above, together with (4.5) and (4.8), we deduce that \( A^{(1,2)}_{R(G), N(G)} \) exists and \( A^{(1,2)}_{R(G), N(G)} = X = G(AG)_{\#} \) by Theorem 3.1.
To show that (iii) implies (i), we use Theorem 2.4 to obtain
\[(AGAA^{(1)})(AGW^2AA^{(1)}) = AGAGW^2AA^{(1)} = AGWAA^{(1)}
= A^{(1,2)}_{R(G), N(G)}AGWAA^{(1)} = A^{(1,2)}_{R(G), N(G)}AA^{(1)}\]
\[= AA^{(1)}.
(4.9)\]
Therefore,
\[(AGAA^{(1)})(AGW^2AA^{(1)})(AGAA^{(1)}) = AA^{(1)}(AGAA^{(1)}) = AGAA^{(1)}\]
and then
\[AG\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right) = 0.\]
By Theorem 3.1, \(R(A) \cap N(G) = \{0\}\) and \(N(A) \cap R(G) = \{0\}\) and so
\[R\left(G\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right)\right) \subset R(G) \cap N(A) = \{0\}.\]
Thus
\[G\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right) = 0.\]
From this, we have
\[R\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right) \subset R(A) \cap N(G) = \{0\},\]
and then
\[(4.10) \quad (AGW^2AA^{(1)})(AGAA^{(1)}) = AA^{(1)}.\]
By (4.9) and (4.10), \(AGAA^{(1)}\) is invertible in \(AA^{(1)}M_m(R)AA^{(1)}\) and so is \(U\) in \(M_m(R)\). Also, obviously, \(A\) is von Neumann regular.

Now we shall prove that (4.1) \(\sim\) (4.3). Since
\[G(AG)^{\|} = G(AV^{-2}G) = GU^{-1}AV^{-1}G = (GU^{-2}A)G = (GA)^{\|}G,\]
we have \(A^{(1,2)}_{R(G), N(G)} = (GA)^{\|}G\) and (4.4).

Next we will prove (4.2). Since
\[GAG = GAG\left((AG)^{\|}\right)^2AGAG,\]
\(GAG\) is von Neumann regular and then
\[AG = U^{-1}UAG = U^{-1}AGAG = (U^{-1}A)GAG(GAG)^{(1)}GAG\]
\[= AG(GAG)^{(1)}GAG.\]
Therefore
\[ A \left( G - G(GAG)^{(1)}GAG \right) = 0. \]

Thus
\[ R \left( G - G(GAG)^{(1)}GAG \right) \subset N(A) \cap R(G) = \{0\}. \]

So we obtain
\[ G = G(GAG)^{(1)}GAG \]

Since \( A^{(1,2)}_{R(G),N(G)} \) exists, using (2.4) and (4.11), it follows that
\[ G = GAGW = GAG(GAG)^{(1)}GAGW \]

(4.12)

Let \( Z = G(GAG)^{(1)}G \). Using (4.11) and (4.12), it easily follows that \( AZA = A, R(Z) = R(G) \) and \( N(Z) = N(G) \). By Theorem 3.1 we have that
\[ A^{(1,2)}_{R(G),N(G)} = Z = G(GAG)^{(1)}G \].

Finally, we will verify (4.3). It is obvious that \( AG \) and \( GA \) are von Neumann regular. By Proposition 4.1 and the invertibility of \( V \) there exists a matrix \( P \in A^{(1)}AM_n(R)A^{(1)}A \) such that \( P(A^{(1)}AGA) = A^{(1)}A \). Thus
\[ A = A\left( PA^{(1)}AGA \right) = AP A^{(1)}A \left( GA(GA)^{(1)}GA \right) = A(GA)^{(1)}GA. \]

Using (4.13), we deduce that \( (AG)^{(1)}A(GA)^{(1)} \) is a \{1\} inverse of \( GAG \). Therefore, using (4.2), we obtain (4.3).

**Remark 2.** By (4.4), we can compute \( A^{(1,2)}_{R(G),N(G)} \) using \( U \) or \( V \).

**Remark 3.** If \( G = A \) where \( A \) is such that \( V = A^{(1)}A^2 + I_n - A^{(1)}A \) is invertible, then \( N(A) \cap R(A) = \{0\} \). Indeed, let \( x \in N(A) \cap R(A) \). Then there exists a \( y \in R^n \) such that \( x = Ay \) and so \( A^2y = 0 \). Since \( V \) is invertible, there exists a matrix \( P \) such that \( PV = I_n \). Thus \( PA^{(1)}A^3 = A^{(1)}A \) and then
\[ 0 = PA^{(1)}A^3y = A^{(1)}Ay. \]

Hence \( Ay = AA^{(1)}Ay = 0 \). Consequently, \( x = Ay = 0 \).

Similarly, if we take \( G = A^* \), where * is an involution on the matrices over \( R \) such that \( U = AA^*A^{(1)}A + I_n - AA^*A^{(1)} \) is invertible, then \( N(A^*) \cap R(A^*) = \{0\} \). Indeed, let \( x \in N(A) \cap R(A^*) \). Then there exists a \( y \in R^n \) such that \( x = A^*y \) and so \( AA^*y = 0 \). Since \( U \) is invertible, there exists a matrix \( Q \) such that \( AA^*AA^{(1)}Q = AA^{(1)} \) and thus
\[ 0 = Q^*(A^{(1)})^*A^*AA^*y = (A^{(1)})^*A^*y. \]

So \( x = A^*y = (A^{(1)})^*A^*y = 0. \)
When $G$ takes the value $A$ (respectively $A^*$) in the theorem above, we find that $A_{R(G),N(G)}^{(1,2)}$ is $A_g$ (respectively $A^*$).

**Theorem 4.4.** Let $A$ be an $m \times n$ matrix over $R$. Then

(i) $A_{R(A),N(A)}^{(1,2)}$ exists if and only if $A_g$ exists. Moreover, $A_{R(A),N(A)}^{(1,2)} = A_g$.

(ii) If $*$ is an involution on the matrices over $R$ then $A_{R(A^*),N(A^*)}^{(1,2)}$ exists if and only if $A^*$ exists. Moreover, $A_{R(A^*),N(A^*)}^{(1,2)} = A^*$.

**Proof.** To show the existence of $A_{R(A),N(A)}^{(1,2)}$ implies existence of $A_g$ in (i), take $G = A$ in (4.1). Then $A_{R(A),N(A)}^{(1,2)} = A(A^2)_g = (A^2)_gA$ and then $AA_{R(A),N(A)}^{(1,2)} = A_{R(A),N(A)}^{(1,2)}A$. Hence $A_{R(A),N(A)}^{(1,2)}$ is the group inverse of $A$.

To show that existence of $A_{R(A^*),N(A^*)}^{(1,2)}$ implies existence of $A^*$ in (ii), take $G = A^*$ in (4.1). Then $A_{R(A^*),N(A^*)}^{(1,2)} = A^*(AA^*)_g = (A^*A)_gA^*$ and then

$$\left( A_{R(A),N(A)}^{(1,2)} \right)^* = AA_{R(A),N(A)}^{(1,2)}$$

and

$$\left( A_{R(A^*),N(A^*)}^{(1,2)} \right)^* = A_{R(A^*),N(A^*)}^{(1,2)}A.$$ 

Hence $A_{R(A^*),N(A^*)}^{(1,2)}$ is the Moore-Penrose inverse of $A$.

The converses follow from Theorem 2.5.

By Theorems 4.3 and 4.4 and Remark 3, we can obtain the following two corollaries, in which the first is equivalent to [8, Corollary 2], and the second is almost the same as [6, Theorem 1].

**Corollary 4.5.** Let $A \in R^{m \times n}$. The following conditions are equivalent.

(i) $A$ is von Neumann regular and $U = A^3A^{(1)} + I_n - AA^{(1)}$ is invertible.

(ii) $A$ is von Neumann regular and $V = A^{(1)}A^3 + I_n - A^{(1)}A$ is invertible.

(iii) $A_g$ exists.

Moreover,

\begin{align*}
A_g &= A(A^2)_g = (A^2)_gA \\
A &= A(A^3)^{(1)}A \\
A_g &= A(A^2)^{(1)}A(A^2)^{(1)}A. \\
A_v &= AU^{-1}A^2 = AU^{-1}AV^{-1}A = A^2V^{-2}A.
\end{align*}

**Remark 4.** The above corollary is unlike [8, Corollary 2], but they are equivalent. This is because $V$ is invertible if and only if $T = A^{(1)}A^2 + I_n - A^{(1)}A$ is invertible. Indeed, if $V$ is invertible, then there exists a matrix $P \in M_n(R)$ such that $PV = VP = I_n$. From this and $V = T^2$, we get $(PT)T = T(TP) = I_n$. Hence $T$ is invertible in $M_n(R)$. The converse is obvious from $V = T^2$. 


**Corollary 4.6.** Let $A$ be an $m \times n$ matrix over $R$ and let $\ast$ be an involution on the matrices over $R$. The following conditions are equivalent.

(i) $A$ is von Neumann regular and $U = AA^*AA^{(1)} + I_n - AA^{(1)}$ is invertible.

(ii) $A$ is von Neumann regular and $V = A^{(1)}AA^*A + I_n - A^{(1)}A$ is invertible.

(iii) $A^+$ exists.

Moreover,

$$A^+ = A^*(AA^*)_g = (A^*A)_gA^* = A^*(A^*AA^*)^{(1)}A^* = A^*(AA^*)^{(1)}A(A^*A)^{(1)}A^* = A^*U^{-2}AA^* = A^*U^{-1}AV^{-1}A^* = A^*AV^{-2}A^*.$$ 

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**References**


The generalized inverse $A^{(2)}_{T,S}$

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