A NOTE ON NORMALITY AND SHARED VALUES
MINGLIANG FANG and LAWRENCE ZALCMAN

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Abstract

Let \( k \) be a positive integer and \( b \) a nonzero constant. Suppose that \( \mathcal{F} \) is a family of meromorphic functions in a domain \( D \). If each function \( f \in \mathcal{F} \) has only zeros of multiplicity at least \( k + 2 \) and for any two functions \( f, g \in \mathcal{F} \), \( f \) and \( g \) share 0 in \( D \) and \( f^{(k)} \) and \( g^{(k)} \) share \( b \) in \( D \), then \( \mathcal{F} \) is normal in \( D \). The case \( f \neq 0, f^{(k)} \neq b \) is a celebrated result of Gu.


Keywords and phrases: meromorphic function, normality, shared value.

1. Introduction

Let \( D \) be a domain in \( \mathbb{C} \) and \( \mathcal{F} \) a family of meromorphic functions defined in \( D \). \( \mathcal{F} \) is said to be normal in \( D \), in the sense of Montel, if each sequence \( \{f_n\} \subset \mathcal{F} \) has a subsequence \( \{f_{n_i}\} \) which converges spherically locally uniformly in \( D \), to a meromorphic function or \( \infty \) (see Hayman [5], Schiff [7], Yang [11]).

Suppose that \( f, g \) are meromorphic functions on \( D \) and \( a \in \mathbb{C} \cup \{\infty\} \). If \( f(z) = a \) if and only if \( g(z) = a \), we say that \( f \) and \( g \) share \( a \) in \( D \).

In 1912, Montel [6] proved the following well-known normality criterion.

**Theorem A.** Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \), and let \( a, b, \) and \( c \) be three distinct values in the extended complex plane. If for each function \( f \in \mathcal{F} \), \( f \neq a, b, c \), then \( \mathcal{F} \) is normal in \( D \).

In 1994, Sun [8] extended Theorem A as follows (see for example [9]).

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**Theorem B.** Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \), and let \( a, b \) and \( c \) be three distinct values in the extended complex plane. If each pair of functions \( f \) and \( g \) in \( \mathcal{F} \) share \( a, b \) and \( c \) in \( D \), then \( \mathcal{F} \) is normal in \( D \).

In 1979, Gu [2] proved the following result.

**Theorem C.** Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \), and let \( k \) be a positive integer and \( b \) a nonzero constant. If for each function \( f \) in \( \mathcal{F} \), \( f \not\equiv 0 \) and \( f^{(k)} \not\equiv b \) in \( D \), then \( \mathcal{F} \) is normal in \( D \).

It is natural to ask whether Theorem C can be extended in the same way that Theorem B extends Theorem A. In this note, we offer such an extension. In each of the results below, \( k \) is a positive integer and \( b \) is a nonzero complex constant.

**Theorem 1.** Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \), all of whose zeros have multiplicity at least \( k + 2 \). If each pair of functions \( f \) and \( g \) in \( \mathcal{F} \) share \( 0 \) in \( D \) and \( f^{(k)} \) and \( g^{(k)} \) share \( b \) in \( D \), then \( \mathcal{F} \) is normal in \( D \).

**Example 1.** Let \( n, k \) be positive integers. Let \( D = \{ z : |z| < 1 \} \) and \( \mathcal{F} = \{ f_n \} \), where
\[
f_n(z) = \frac{n^{k+1}}{k!(nz - 1)}, \quad n = 1, 2, 3, \ldots
\]
Each function in \( \mathcal{F} \) has a single zero of multiplicity \( k + 1 \). Clearly, for each pair \( m, n \) of positive integers, \( f_m, f_n \) share \( 0 \) in \( D \). Moreover, since
\[
f_n(z) = \frac{1}{k!} \left( z^k + \frac{1}{n} z^{k-1} + \cdots + \frac{1}{n^{k-1}} z + \frac{1}{n^k} + \frac{1}{n^{k+1}} \right),
f_n^{(k)}(z) = 1 + \frac{(-1)^k}{(nz - 1)^{k+1}} \not= 1.
\]
Thus \( f_m^{(k)} \) and \( f_n^{(k)} \) also share the value 1 in \( D \). But \( \mathcal{F} \) clearly fails to be normal on any neighbourhood of 0. This shows that the condition in Theorem 1 that the zeros of functions in \( \mathcal{F} \) have multiplicity at least \( k + 2 \) cannot be weakened.

**Theorem 2.** Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \), all of whose zeros have multiplicity at least \( k + 1 \) and whose poles have multiplicity at least 2. If each pair of functions \( f \) and \( g \) in \( \mathcal{F} \) share \( 0 \) in \( D \) and \( f^{(k)} \) and \( g^{(k)} \) share \( b \) in \( D \), then \( \mathcal{F} \) is normal in \( D \).

**Corollary 3.** Let \( \mathcal{F} \) be a family of holomorphic functions defined in \( D \), all of whose zeros have multiplicity at least \( k + 1 \). If each pair of functions \( f \) and \( g \) in \( \mathcal{F} \) share \( 0 \) in \( D \) and \( f^{(k)} \) and \( g^{(k)} \) share \( b \) in \( D \), then \( \mathcal{F} \) is normal in \( D \).
Corollary 4. Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$. If each pair of functions $f$ and $g$ in $\mathcal{F}$ share $0$ in $D$ and $f^m f'$ and $g^n g'$ share $b$ in $D$, then $\mathcal{F}$ is normal in $D$.

To prove Corollary 4, set $\mathcal{F} = \{f^{m+1}/(m+1) : f \in \mathcal{F}\}$ and apply Theorem 2 to this family with $k = 1$.

Example 2. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where $f_n(z) = nz^k$, $n = 1, 2, 3, \ldots$. Then the zeros of functions in $\mathcal{F}$ all have multiplicity $k$. Moreover, any pair of functions $f$ and $g$ in $\mathcal{F}$ clearly share $0$ in $D$ and $f^{(k)}$ and $g^{(k)}$ share $1/2$ in $D$; but $\mathcal{F}$ is not normal in $D$. This shows that the condition that the zeros of functions in $\mathcal{F}$ have multiplicity at least $k + 1$ in Theorem 2 and Corollary 3 is best possible.

2. Some lemmas

For the proofs of Theorem 1 and Theorem 2, we require the following results.

Lemma 1 ([9, Theorem 7]). Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k + 2$. If $f^{(k)} \neq b$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in $D$.

Lemma 2 ([9, Theorem 5]). Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k + 1$ and whose poles have multiplicity at least $2$. If $f^{(k)} \neq b$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in $D$.

Below, we assume the basic results and notation of Nevanlinna Theory [4, 12]. In particular, $S(r, f)$ denotes any function satisfying $S(r, f) = O(\log r T(r, f))$ as $r \to \infty$, possibly outside a set of finite measure, where $T(r, f)$ is Nevanlinna’s characteristic function. In fact, the functions for which we use this notation are all of finite order, so the exceptional set does not occur. For such functions, we have $S(r, f) = o(T(r, f))$ [4, page 41].

Lemma 3 ([4, Theorem 3.2]). Let $f$ be a nonconstant meromorphic function in the complex plane. Then

$$T(r, f) \leq N(r, f) + N(r, 1/f) + N(r, 1/(f^{(k)} - b)) + S(r, f).$$

By [4, page 61], we also have

Lemma 4. Let $f$ be a nonconstant meromorphic function in the complex plane. Then

$$\overline{N}(r, f) \leq \left(1 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(1 + \frac{2}{k}\right) \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f).$$
Lemma 5. Let $f$ be a meromorphic function in the complex plane and $l$ a positive integer satisfying $l > k + 4 + 2/k$. If $f \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least $l$, then $f$ is a constant.

Proof. Since $f \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least $l$, we have by (2.2)

$$\bar{N}(r, f) \leq \left(1 + \frac{2}{k}\right) \bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f)$$

$$\leq \frac{1 + 2/k}{l} \bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f)$$

$$\leq \frac{1 + 2/k}{l} T\left(r, f^{(k)}\right) + S(r, f)$$

$$\leq \frac{1 + 2/k}{l} [T(r, f) + k\bar{N}(r, f)] + S(r, f).$$

Thus by (2.3) we get

$$\bar{N}(r, f) \leq \frac{k + 2}{k(l - k - 2)} T(r, f) + S(r, f).$$

By (2.1) and the facts that $f \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least $l$, we have

$$T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f)$$

$$\leq \bar{N}(r, f) + \frac{1}{l} \bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f)$$

$$\leq \bar{N}(r, f) + \frac{1}{l} T\left(r, f^{(k)}\right) + S(r, f)$$

$$\leq \bar{N}(r, f) + \frac{1}{l} [T(r, f) + k\bar{N}(r, f)] + S(r, f)$$

$$\leq \left(1 + \frac{k}{l}\right) \bar{N}(r, f) + \frac{1}{l} T(r, f) + S(r, f).$$

Thus

$$T(r, f) \leq \frac{l + k}{l - 1} \bar{N}(r, f) + S(r, f).$$

By (2.4) and (2.6), we have

$$T(r, f) \leq \frac{(k + 2)(l + k)}{k(l - 1)(l - k - 2)} T(r, f) + S(r, f).$$
that is, 
\[(l-1)(l-k-2)-(k+2)(l+k)]T' = S(r, f).
\] Since \( l > k + 4 + 2/k \), we have 
\[(l-1)(l-k-2)-(k+2)(l+k) > 0.\]
Thus 
\[T(r, f) = S(r, f),\]
so \( f \) is constant.

**Lemma 6 ([3, Theorem 3], [4, Corollary to Theorem 3.5]).** Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C} \), and let \( b \) be a nonzero value. Then for each positive integer \( k \), either \( f \) or \( f^{(k)} - b \) vanishes. If \( f \) is transcendental, then for each positive integer \( k \), either \( f \) or \( f^{(k)} - b \) has infinitely many zeros.

**Lemma 7 ([10, 13]).** Let \( \mathcal{F} \) be a family of functions meromorphic on the unit disc. Suppose that each \( f \in \mathcal{F}, f \neq 0. \) Then if \( \mathcal{F} \) is not normal, there exist, for each \( \alpha \geq 0 \),

- (a) a number \( 0 < r < 1 \);
- (b) points \( z_n, |z_n| < r \);
- (c) functions \( f_n \in \mathcal{F} \); and
- (d) positive numbers \( \rho_n \to 0 \)

such that \( \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta) \) locally uniformly with respect to the spherical metric, where \( g \) is a nonconstant meromorphic function on \( \mathbb{C} \).

### 3. Proof of Theorem 1

**Proof of Theorem 1.** Let \( z_0 \in D \). We show that \( \mathcal{F} \) is normal at \( z_0 \). Let \( f \in \mathcal{F} \).

We consider two cases.

**Case 1:** \( f^{(k)}(z_0) \neq b \). Then there exists a disk \( D_\delta = \{ z : |z - z_0| < \delta \} \) such that \( f^{(k)} \neq b \) in \( D_\delta \). Thus, for every \( g \in \mathcal{F} \), the zeros of \( g \) have multiplicity at least \( k + 2 \) and \( g^{(k)} \neq b \) in \( D_\delta \). By Lemma 1, \( \mathcal{F} \) is normal in \( D_\delta \). Hence \( \mathcal{F} \) is normal at \( z_0 \).

**Case 2:** \( f^{(k)}(z_0) = b \). Then, by the condition of the theorem, \( f(z_0) \neq 0 \). Hence there exists a disk \( D_\delta = \{ z : |z - z_0| < \delta \} \) such that \( f \neq 0 \) in \( D_\delta \) and \( f^{(k)} \neq b \) in \( D_\delta = \{ z : 0 < |z - z_0| < \delta \} \). Hence, by Lemma 1, \( \mathcal{F} \) is normal in \( D_\delta \). We complete the proof of the theorem by using the method of Yang [11].

Let \( \{ f_n \} \) be a sequence in \( \mathcal{F} \); then there exists a subsequence of \( \{ f_n \} \) (which, without loss of generality, we may again denote by \( \{ f_n \} \)) which converges locally spherically uniformly on \( D_\delta \) to a function \( h \). We consider two subcases.

**Case 2.1:** \( h \neq 0 \). Then, by Hurwitz’s Theorem, \( h \neq 0 \) in \( D_\delta \). Therefore,

\[
\min_{0 < |t| < 2\delta} |h(t + \delta e^{i\theta}/2)| > A > 0
\]

for some constant \( A \).
Hence for sufficiently large \( n \),
\[
\min_{0 \leq \theta \leq 2\pi} \left| f_n \left( z_0 + \frac{\delta}{2} e^{i\theta} \right) \right| > \frac{A}{2} > 0.
\]

Since \( f_n \) is meromorphic and \( f_n \neq 0 \) in \( D_{\delta} \), \( 1/f_n \) is holomorphic in \( D_{\delta/2} = \{ z : |z - z_0| \leq \delta/2 \} \), and
\[
\max_{0 \leq \theta \leq 2\pi} \left| f_n(z_0 + \delta e^{i\theta}/2) \right| < \frac{2}{A}.
\]

By the maximum principle, we conclude that
\[
\max_{|z - z_0| \leq \delta/2} \left| \frac{1}{f_n(z)} \right| < \frac{2}{A},
\]
so
\[
\min_{|z - z_0| \leq \delta/2} |f_n(z)| > \frac{A}{2} > 0.
\]

Hence there exists a subsequence of \( \{ f_n \} \) which converges locally spherically uniformly in \( D_{\delta/2} \).

**Case 2.2: \( h \equiv 0 \).** Then \( \{ f_n \} \) converges locally uniformly to 0 in \( D_{\delta} \). Thus \( \{ f^{(k)}_n \} \) and \( \{ f^{(k+1)}_n \} \) also converge locally uniformly to 0. Hence, for sufficiently large \( n \), we have by the argument principle

\[
N \left( \frac{\delta}{2}, z_0, f^{(k)}_n - b \right) - N \left( \frac{\delta}{2}, z_0, \frac{1}{f^{(k+1)}_n - b} \right) = \left| \frac{1}{2\pi i} \int_{|z - z_0| = \delta/2} \frac{f^{(k+1)}_n(z)}{f^{(k)}_n(z) - b} \, dz \right| < 1.
\]

Thus we have
\[
N \left( \frac{\delta}{2}, z_0, f^{(k)}_n - b \right) = N \left( \frac{\delta}{2}, z_0, \frac{1}{f^{(k)}_n - b} \right).
\]

Since any pole of \( f^{(k)}_n - b \) must have multiplicity at least \( k + 1 \), it follows that the zero of \( f^{(k)}_n - b \) at \( z_0 \) has multiplicity at least \( k + 1 \).

We consider two subcases.

**Case 2.2.1.** The set \( S \) of positive integers \( n \) such that the zeros of \( f^{(k)}_n - b \) at \( z_0 \) have multiplicity greater than \( k + 4 + 2/k \) is infinite. We claim that \( G = \{ f_n : n \in S \} \) is normal in \( D_{\delta/2} \).

Indeed, suppose that \( G \) is not normal in \( D_{\delta/2} \). Then by Lemma 7, we have (renumbering, as we may) \( f_n \in G, z_n \in D_{\delta/2}, \) and \( \rho_n \to 0^+ \) such that
\[
g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n} \to g(\xi)
\]
locally uniformly with respect to the spherical metric, where $g$ is a nonconstant meromorphic function on $\mathbb{C}$.

By Hurwitz’s Theorem, $g \neq 0$ and any zeros of $g^{(k)} - b$ have multiplicity greater than $k + 4 + 2/k$. Thus, by Lemma 5, $g$ is constant, a contradiction. Hence there exists a subsequence of $\{ f_n \}$ which converges locally spherically uniformly in $D_{b/2}$.

**Case 2.2.2.** The set $S_l$ of positive integers $n$ such that the zeros of $f_n^{(k)} - b$ at $z_0$ have multiplicity $l$ for some positive integer $l$ such that $k + 1 \leq l \leq k + 4 + 2/k$ is infinite. We claim that $G = \{ f_n : n \in S_l \}$ is normal in $D_{b/2}$.

In fact, suppose that $G$ is not normal in $D_{b/2}$. Then by Lemma 7, we have (again renumbering) $f_n \in G$, $z_n \in D_{b/2}$, and $\rho_n \to 0^+$ such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^k} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g$ is a nonconstant meromorphic function on $\mathbb{C}$.

By Hurwitz’s Theorem, $g \neq 0$ and each zero of $g^{(k)} - b$ has multiplicity at least $l$. We claim, in addition, that $g^{(k)} - b$ has only a single zero. That $g^{(k)} - b$ must vanish somewhere follows from Lemma 6. Suppose that $\xi_1$ and $\xi_2$ are distinct zeros of $g^{(k)} - b$; then the zeros of $g^{(k)} - b$ at $\xi_1$ and $\xi_2$ have multiplicity at least $l$. Let $\gamma$ be a simple closed curve containing $\xi_1$ and $\xi_2$ in its interior and such that $g$ has no zeros on $\gamma$ and no poles on or inside $\gamma$. Then $g_n(\xi)$ converges to $g(\xi)$ uniformly on and inside $\gamma$, and so $g_n^{(k)} - b$ converges to $g^{(k)} - b$ uniformly on and inside $\gamma$. By the argument principle, $g_n^{(k)} - b$ and $g^{(k)} - b$ have the same number of zeros (counting multiplicity) inside $\gamma$ for sufficiently large $n$. But $g_n^{(k)} - b$ has only $l$ zeros (counting multiplicity) while $g^{(k)}$ has at least $2l$ zeros (counting multiplicity) for sufficiently large $n$, which is a contradiction.

From the above discussion, $g^{(k)} - b$ has only a single zero, whose multiplicity is $l$. Since $f_n^{(k)}(z_n + \rho_n \xi) = g_n^{(k)}(\xi)$, which converges to $g^{(k)}(\xi)$ uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of $g$, it follows from the formula after (3.1) that $f_n^{(k)}$ has $l$ poles (counting multiplicity) in $D_{b/2}$ and hence $g_n^{(k)}$ has $l$ poles (counting multiplicity) on the disc $\{ \xi : z_n + \rho_n \xi \in D_{b/2} \}$. We conclude easily from the argument principle that $g^{(k)}$ has at most $l$ poles (counting multiplicity) in $\mathbb{C}$.

Thus

(i) $g \neq 0$;
(ii) $g^{(k)} - b$ has a single zero, whose multiplicity is $l$;
(iii) $g^{(k)}$ has at most $l$ poles, counting multiplicities.

We claim that no such function exists. By Lemma 6, there is no transcendental function, satisfying (i) and (ii). Clearly, $g$ cannot be a polynomial. We now turn to the somewhat tedious verification that no rational function satisfies conditions (i), (ii), and (iii). We consider three subcases.
**Case 2.2.2.1: k ≥ 3.** Since \(k + 1 \leq l \leq k + 4 + 2/k\), \(g\) has only a single pole. Thus \(g(\xi) = A/(\xi - a_1)^m\), where \(A\) is a nonzero constant, \(a_1\) is a constant, and \(m\) is a positive integer.

Obviously, \(g^{(k)} - b\) has \(m + k\) distinct zeros, which contradicts the fact that \(g^{(k)} - b\) has a single zero.

**Case 2.2.2.2: k = 2.** Since \(3 \leq l \leq 7\), \(g\) has one of the following forms:

1. \(g(\xi) = A/(\xi - a_1)(\xi - a_2)^2, l = 7;\)
2. \(g(\xi) = A/(\xi - a_1)(\xi - a_2), l = 6;\)
3. \(g(\xi) = A/(\xi - a_1)^m, l = m + 2, 1 \leq m \leq 5,\)

where \(A\) is a nonzero constant, \(a_1\) and \(a_2\) are distinct constants, and \(m\) is a positive integer.

If \(g(\xi) = A/[l(\xi - a_1)(\xi - a_2)^2]\), then

\[
g''(\xi) - b = -\frac{A[3(\xi - a_1)(\xi - a_2) - (3\xi - 2a_1 - a_2)(5\xi - 3a_1 - 2a_2)]}{(\xi - a_1)^3(\xi - a_2)^4} - \frac{b(\xi - a_1)^3(\xi - a_2)^4}{(\xi - a_1)^3(\xi - a_2)^4}.
\]

Since \(g'' - b\) has only a single zero, we have

\[
(3.2) \quad A[3(\xi - a_1)(\xi - a_2) - (3\xi - 2a_1 - a_2)(5\xi - 3a_1 - 2a_2)] + b(\xi - a_1)^3(\xi - a_2)^4 = b(\xi - c)^7.
\]

Differentiating the two sides of (3.2) three times, we have

\[
(3.3) \quad (\xi - a_2)p(\xi) = 210b(\xi - c)^4,
\]

where \(p\) is a polynomial and \(c\) is a constant.

Thus \(a_2 = c\). It then follows from (3.2) that \(a_1 = a_2\), a contradiction.

If \(g\) is of the form (2) or (3), we can similarly get a contradiction.

**Case 2.2.2.3: k = 1.** Since \(2 \leq l \leq 7\), \(g\) has one of the following forms:

1. \(g(\xi) = A/(\xi - a_1)(\xi - a_2)^2, l = 7;\)
2. \(g(\xi) = A/(\xi - a_1)(\xi - a_2)(\xi - a_3), l = 6;\)
3. \(g(\xi) = A/(\xi - a_1)^2(\xi - a_2)^3, l = m + 4, 2 \leq m \leq 3;\)
4. \(g(\xi) = A/(\xi - a_1)(\xi - a_2)^m, l = m + 3, 1 \leq m \leq 4;\)
5. \(g(\xi) = A/(\xi - a_1)^m, l = m + 1, 1 \leq m \leq 6,\)

where \(A\) is a nonzero constant, \(a_1\), \(a_2\) and \(a_3\) are distinct constants, and \(m\) is a positive integer.
We deal with case (1). If 
\[ g \circ D^A = T. \]
then
\[ g - b = -A[(2\xi - a_1 - a_2)(\xi - a_3) + 2(\xi - a_1)(\xi - a_2)]/\]
\[ (\xi - a_1)(\xi - a_2)^2(\xi - a_3)^3. \]

Since \( g' - b \) has only a single zero, we have
\[ A[(2\xi - a_1 - a_2)(\xi - a_3) + 2(\xi - a_1)(\xi - a_2)] \]
\[ + b(\xi - a_1)^2(\xi - a_2)^2(\xi - a_3)^3 = b(\xi - c)^7. \]

Differentiating the two sides of (3.4), we have
\[ b(\xi - a_1)(\xi - a_2)(\xi - a_3)^2[2(2\xi - a_1 - a_2)(\xi - a_3) + 3(\xi - a_1)(\xi - a_2)] \]
\[ + A(8\xi - 3a_1 - 3a_2 - 2a_1) = 7b(\xi - c)^6. \]

Setting \( \xi = a_3 \) in (3.5) gives
\[ 3A(2a_3 - a_1 - a_2) = 7b(a_3 - c)^6. \]

Differentiating the two sides of (3.5), we obtain
\[ 8A + (\xi - a_3)p(\xi) = 42b(\xi - c)^5, \]
where \( p \) is a polynomial.

Setting \( \xi = a_3 \) in (3.7), we get
\[ 8A = 42b(a_3 - c)^5. \]

Thus by (3.6) and (3.8) we have
\[ c = -\frac{7}{2}a_3 + \frac{9}{4}a_1 + \frac{9}{4}a_2. \]

On the other hand, differentiating both sides of (3.4) six times and putting \( \xi = c \), we obtain
\[ c = (2a_1 + 2a_2 + 3a_3)/7. \]

Comparing (3.9) and (3.10) gives \( a_3 = c \), which contradicts (3.8) since \( A \neq 0 \).

If \( g \) has one of the other forms, we obtain a contradiction in a similar fashion.

Thus we have proved that \( \{f_n\} \) is normal in \( D_{3/2} \). Hence, there exists a subsequence of \( \{f_n\} \) which converges locally spherically uniformly in \( D_{3/2} \). It follows that \( \mathcal{F} \) is normal at \( z_0 \), and so \( \mathcal{F} \) is normal in \( D \). The proof of the theorem is complete.\( \square \)
The proof of Theorem 2, which uses Lemma 2, is similar. We omit the details.

References


Department of Mathematics              Department of Mathematics and Statistics
Nanjing Normal University            Bar-Ilan University
Nanjing 210097                       52900 Ramat-Gan
P. R. China                         Israel
e-mail: mlfang@pine.njnu.edu.cn       e-mail: zalcman@macs.biu.ac.il