HIGHER DIMENSIONAL COHOMOLOGY OF WEIGHTED SEQUENCE ALGEBRAS

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Abstract

It is well known that $c_0$ is amenable and so its global dimension is zero. In this paper we will investigate the cyclic and Hochschild cohomology of Banach algebra $c_0$, and its unitisation with coefficients in its dual space, where $\omega$ is a weight on $\mathbb{Z}$ which satisfies $\inf(\omega(i)) = 0$. Moreover we show that the weak homological bi-dimension of $c_0$ is infinity.


1. Introduction

The Banach algebra $\mathcal{A}$ is amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{B}) = 0$ for every Banach $\mathcal{A}$-bimodule $\mathcal{B}$. This definition was introduced by Johnson in (1972) [8]. The Banach algebra $\mathcal{A}$ is weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A'}) = 0$. This definition generalizes the one which was introduced by Bade, Curtis and Dales in [1], where it was noted that a commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if $\mathcal{H}^1(\mathcal{A}, \mathcal{B}) = 0$ for every symmetric Banach $\mathcal{A}$-bimodule $\mathcal{B}$.

Johnson in [8] proved that for an amenable Banach algebra $\mathcal{A}$, the cohomology groups $\mathcal{H}^n(\mathcal{A}, \mathcal{B})$ vanish for every Banach $\mathcal{A}$-bimodule $\mathcal{B}$ and all $n \geq 1$. The question was raised whether in general $\mathcal{H}^n(\mathcal{A}, \mathcal{A'}) = 0$ for a weakly amenable Banach algebra $\mathcal{A}$ and all $n \geq 1$. The question was answered in the negative in [14] by showing that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^\infty(\mathbb{F}_2)) \neq 0$. In fact Johnson [8] showed that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C}) \neq 0$ and in [14] Sinclair and Smith showed that the non-trivial cohomology group $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C})$ is naturally embedded as a direct summand of...
In this paper we will give an example of a weakly amenable Banach algebra, such that the $n$th cohomology groups with coefficients in the dual space do not vanish for all $n > 1$.

It is a question of general interest whether or not the $n$th cohomology group is necessarily zero. This, and closely related questions have stimulated much of the recent development of the theory of cohomology groups.

Bade, Curtis and Dales in [1] showed that $\mathcal{H}^2(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)) \neq 0$. This may lead one to believe that $\mathcal{H}^n(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+))$ for all $n \geq 2$ are also non-zero. However, Johnson showed in [10] that the alternating cohomology of $\ell^1(\mathbb{Z}_+)$ vanishes in all dimensions strictly greater than one. Then Dales and Duncan [2, Theorem 3.2] showed that $\mathcal{H}^2(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)) = 0$. Gourdeau and White in [4] with a complicated proof showed that $\mathcal{H}^3(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)) = 0$. This leads to the conjecture that all the cohomology groups of $\ell^1(\mathbb{Z}_+)$ with coefficients in $\ell^1(\mathbb{Z}_+)$ vanish for $n > 3$.

In this paper for the weakly amenable Banach algebra $\mathcal{A}$, the unitisation of $\mathcal{A}$, we show that the cyclic cohomology group $\mathcal{H}^n(\mathcal{A}, (\mathcal{A})')$ and the Hochschild cohomology group $\mathcal{H}^n(\mathcal{A}, (\mathcal{A})')$ are non-trivial for every $n \geq 2$.

Let $\omega$ be a weight sequence on $\mathbb{Z}$, that is, $\omega$ is a non-zero, positive valued function on $\mathbb{Z}$ such that $\omega(n) \leq 1$ for every $n \in \mathbb{Z}$. Set

$$c_0(\mathbb{Z}, \omega^{-1}) = \{ a = \{a_n\} : n \in \mathbb{Z}, \lim_{|n| \to \infty} \frac{|a_n|}{\omega(n)} = 0 \},$$

where $c_0(\mathbb{Z}, \omega^{-1})$ is a closed subalgebra of

$$\ell^\infty(\mathbb{Z}, \omega^{-1}) = \{ a = \{a_n\} : n \in \mathbb{Z}, \|a\|_{\omega^{-1}} = \sup \left\{ \frac{|a_n|}{\omega(n)} : n \in \mathbb{Z} \right\} < \infty \}$$

and $c_0(\mathbb{Z}, \omega^{-1})'$ (the dual space of $c_0(\mathbb{Z}, \omega^{-1})$) is equal to

$$\ell^1(\mathbb{Z}, \omega) = \{ a = \{a_n\} : n \in \mathbb{Z}, \sum_{n=-\infty}^{\infty} |a_n|\omega(n) < \infty \}.$$

The element $e_i = \{\delta_{ij}\}_{j \in \mathbb{Z}}, i \in \mathbb{Z}$ is an idempotent, where $\delta_{ij}$ denotes the Kronecker delta. We denote the linear span of such elements by $E$, which is a dense subset of $c_0(\mathbb{Z}, \omega^{-1})$; since if $a \in c_0(\mathbb{Z}, \omega^{-1})$, then we define

$$a^n = \sum_{i=-n}^{n} a_i e_i = \{ \ldots, 0, a_{-n}, \ldots, a_n, 0, \ldots \}$$

and

$$\|a - a^n\|_{\omega^{-1}} = \sup_{|i|>|n|} \frac{|a_i|}{\omega(n)} \to 0 \quad \text{as} \quad |n| \to \infty.$$
Since a commutative Banach algebra which is the closed linear span of its idempotents is weakly amenable \[9\], then \(c_0(\mathbb{Z}, \omega^{-1})\) is weakly amenable, and by \[3, Proposition 1.4\] \(\mathfrak{A}^a\), the unitisation of \(\mathfrak{A} = c_0(\mathbb{Z}, \omega^{-1})\) is also weakly amenable.

**NOTE.** In this paper every weight \(\omega\) on \(\mathbb{Z}\) which we consider must satisfy the condition \(\inf\{\omega(i)\} = 0\), because if \(\inf\{\omega(i)\} \neq 0\), then \(\omega^{-1}\) is a bounded weight and so \(c_0(\mathbb{Z}, \omega^{-1}) \cong c_0(\mathbb{Z})\) which is amenable.

Throughout \(\mathfrak{A}^a\) means the unitisation of \(\mathfrak{A} = c_0(\mathbb{Z}, \omega^{-1})\). Let \(1\) be the unit element of \(\mathfrak{A}^a\). Suppose \(E_N\) is the closed linear span of \(\{e_i\}_{i=1}^N\). Then \(E_N\) is a closed subalgebra of \(\mathfrak{A}^a\). If \(a \in \mathfrak{A}^a\), then \(a = a' + \alpha 1\), where \(a' = \{a_n\}_{n \in \mathbb{Z}}\) is in \(c_0(\mathbb{Z}, \omega^{-1})\) and \(\alpha \in \mathbb{C}\). The norm on \(\mathfrak{A}^a\) is defined by \(\|a\|_{\omega^{-1}} = \|a'\|_{\omega^{-1}} + |\alpha|\). Also for every \(a = a' + \alpha 1\) and \(b = b' + \beta 1\) in \(\mathfrak{A}^a\) we define \(ab = a'b' + \alpha b' + \beta a' + \alpha \beta 1\). Clearly \(E_N \cong \mathbb{C}^N\) and since a direct sum of amenable algebras is amenable, then \(E_N\) is an amenable closed subalgebra of \(\mathfrak{A}^a\).

Note that for every \(\phi \in \mathcal{Z}^n(\mathfrak{A}^a, (\mathfrak{A}^a)^\prime)\), the space of all bounded \(n\)-cocycles, by \[11\] there exists \(\psi_N\) in \(\mathcal{G}^{n-1}(\mathfrak{A}^a, (\mathfrak{A}^a)^\prime)\) such that

\[(\phi - \delta \psi_N)(a_1, \ldots, a_n) = 0 \quad \text{if any one of } a_1, \ldots, a_n \text{ lies in } E_N.\]

But we will show that this is not true for the whole of \(\mathfrak{A}^a\), in fact for every \(n \geq 2\) we will find a (cyclic) cocycle \(\phi \in \mathcal{Z}^n(\mathfrak{A}^a, (\mathfrak{A}^a)^\prime)\) which does not co-bound.

The weak homological bi-dimension of a Banach algebra \(\mathfrak{A}\), denoted by \(wdb\mathfrak{A}\), is the smallest integer \(n\) such that \(\mathcal{H}^m(\mathfrak{A}, X) = 0\) for all Banach \(\mathfrak{A}\)-bimodules \(X\) and all \(m \geq n\), or \(wdb\mathfrak{A} = \infty\) if there is no such \(n\). If \(\mathfrak{A}\) is an amenable Banach algebra, then \(wdb\mathfrak{A} = 0\) \[7, Section 2.5\]. The weak homological bi-dimension of a Banach algebra is a number that measures how much this algebra is homologically worse than amenable. The homological bi-dimension of a Banach algebra \(\mathfrak{A}\), denoted by \(db\mathfrak{A}\), is the smallest integer \(n\) such that \(\mathcal{H}^m(\mathfrak{A}, X) = 0\) for all Banach \(\mathfrak{A}\)-bimodules \(X\) and all \(m \geq n\), or \(db\mathfrak{A} = \infty\) if there is no such \(n\). For every Banach algebra \(\mathfrak{A}\), we have \(wdb\mathfrak{A} \leq db\mathfrak{A}\) (see \[7, VII, Section 3.4\] and \[13\]).

A consequence of the main results of this paper (Theorem 2.2 and Theorem 3.4) is that the weak homological bi-dimension of \(c_0(\mathbb{Z}, \omega^{-1})\) is infinity, that is,

\[wdb c_0(\mathbb{Z}, \omega^{-1}) = \infty.\]

The paper is organized as follows. In Section 2 we calculate the even dimensional cyclic and Hochschild cohomology groups of \(\mathfrak{A}^a\) with coefficients in \((\mathfrak{A}^a)^\prime\), the dual space of \(\mathfrak{A}^a\). In Section 3 we will continue our argument for the odd dimensional case.
2. Even dimensional cohomology groups of weighted sequence algebras

In this section we prove that $\mathcal{H}^{2n}(\mathcal{O}^{\mathbb{Z}}_\mathbb{R}, (\mathcal{O}^{\mathbb{Z}}_\mathbb{R})') \neq 0$ and $\mathcal{H}^{\mathbb{Z}}\mathcal{C}^{2n}(\mathcal{O}^{\mathbb{Z}}_\mathbb{R}) \neq 0$ for every $n \in \mathbb{N}$.

LEMMA 2.1. Let $\sum_{i=-\infty}^{\infty} \alpha_i$ be an absolutely convergent series of real numbers, and let

$$\phi : \mathcal{O}^{\mathbb{Z}}_\mathbb{R} \times \mathcal{O}^{\mathbb{Z}}_\mathbb{R} \times \cdots \times \mathcal{O}^{\mathbb{Z}}_\mathbb{R} \to (\mathcal{O}^{\mathbb{Z}}_\mathbb{R})'$$

be the function defined by

$$\phi(a_1, \ldots, a_{2n})(a_{2n+1}) = \sum_{i=-\infty}^{\infty} \frac{a'_{i} \cdots a'_{(2n+1)i}}{\omega(i)^{2n+1}} a_i ,$$

where $a_i = a'_i + \beta_i 1$ and $a'_i = [a'_{i1}],_{\mathbb{Z}} (k = 1, 2, \ldots, 2n + 1).$ Then $\phi$ is a bounded cyclic $2n$-cocycle for every $n \in \mathbb{N}$.

PROOF. It is easy to see that $\phi$ is a $2n$-linear map. Also

$$|\phi(a_1, \ldots, a_{2n})(a_{2n+1})| \leq \sum_{i=-\infty}^{\infty} \frac{|a'_{i} \cdots a'_{(2n+1)i}|}{\omega(i)^{2n+1}} |\alpha_i|$$

$$\leq \sup_i \left( \frac{|a'_i|}{\omega(i)} \right) \cdots \sup_i \left( \frac{|a'_{(2n+1)i}|}{\omega(i)} \right) \left( \sum_{i=-\infty}^{\infty} |\alpha_i| \right)$$

$$\leq \|a_1\| \cdots \|a_{2n+1}\| \left( \sum_{i=-\infty}^{\infty} |\alpha_i| \right) .$$

Thus $\phi$ is bounded and $\|\phi\| \leq \sum_{i=-\infty}^{\infty} |\alpha_i|$. Now we want to show that $\phi$ is a $2n$-cocycle, that is,

$$\delta \phi(a_1, \ldots, a_{2n+1})(a_{2n+2}) = a_1 \phi(a_2, \ldots, a_{2n+1})(a_{2n+2})$$

$$+ \sum_{i=1}^{2n} (-1)^i \phi(a_1, \ldots, a_{i-1}, a_{i+2}, \ldots, a_{2n+1})(a_{2n+2})$$

$$+ (-1)^{2n+1} \phi(a_1, \ldots, a_{2n}) a_{2n+1} \phi(a_{2n+2}) = 0 .$$

Now we calculate all terms on the right-hand side of the above equation and we obtain the following $(2n + 2)$ terms respectively:

$$\sum_{i=-\infty}^{\infty} \frac{\alpha_i}{\omega(i)^{2n+1}} \left( a'_{i1} \cdots a'_{(2n+2)i} + \beta_1 a'_{2i} \cdots a'_{(2n+2)i} + \beta_{(2n+2)} a'_{i1} \cdots a'_{(2n+1)i} \right) .$$
\(- (a'_i \cdots a'_{(2n+2)i}) + \beta_1 a'_i a'_{2i} \cdots a'_{(2n+2)i}) + \beta_2 a'_i a'_{3i} \cdots a'_{(2n+2)i})
\pm \cdots \pm \left( a'_i \cdots a'_{(2n+2)i} + \beta_2 a'_i a'_{(2n+1)i} \cdots a'_{(2n+2)i}
+ \beta_{2n+1} a'_i \cdots a'_{(2n+i)} a'_{(2n+2)i}) - (a'_i \cdots a'_{(2n+2)i}) + \beta_{2n+1} a'_i \cdots a'_{(2n+i)} a'_{(2n+2)i}) \right).\]

So all terms in the above equation cancel in pairs. Thus \( \phi \) is a 2\(n\)-cocycle, and obviously it is cyclic, that is,

\[
\phi(a_1, \ldots, a_{2n})(a_{2n+1}) = (-1)^{2n} \phi(a_2, \ldots, a_{2n}, a_{2n+1})(a_1).
\]

**Theorem 2.2.** Let \( \omega \) be a weight on \( \mathbb{Z} \) such that \( \inf \{\omega(i)\} = 0 \). Then

\[
\mathcal{H}^{2n}(\mathcal{C}^\omega, (\mathcal{C}^\omega)') \neq 0
\]

and also \( \mathcal{H}^n(\mathcal{C}^\omega) \neq 0 \) for every \( n \in \mathbb{N} \).

**Proof.** Let \( \phi \) be the bounded cyclic 2\(n\)-cocycle which was introduced in Lemma 2.1 and let \( \alpha_i \) be defined as below. Since \( \inf \{\omega(i)\} = 0 \), then there exist numbers \( m_k \), \((k = 1, 2, \ldots)\) such that \( m_i \neq m_j \) whenever \( i \neq j \) and \( \omega(m_k) \leq 1/2^k \). Now we define

\[
\alpha_i = \begin{cases} 1/k^2 & \text{if } i = m_k \quad (k = 1, 2, \ldots); \\
0 & \text{otherwise} \end{cases}
\]

and so \( \sum_{i=-\infty}^{\infty} \alpha_i = \sum_{k=1}^{\infty} 1/k^2 \) which converges. Thus by Lemma 2.1

\[
\phi(a_1, \ldots, a_{2n})(a_{2n+1}) = \sum_{k=1}^{\infty} \frac{a'_{1m_k} \cdots a'_{(2n+1)m_k}}{\omega(m_k)2^{n+1}k^2}
\]

is a bounded cyclic 2\(n\)-cocycle for every \( n \in \mathbb{N} \). Now if there exists a \( \psi \) in \( \mathcal{C}^{2n-1}((\mathcal{C}^\omega, (\mathcal{C}^\omega)')) \) such that

\[
\phi(a_1, \ldots, a_{2n})(a_{2n+1}) = 2 \psi(a_1, \ldots, a_{2n})(a_{2n+1})
= a_1 \psi(a_2, \ldots, a_{2n})(a_{2n+1})
+ \sum_{i=1}^{2n-1} (-1)^i \psi(a_1, \ldots, a_{2i-1}, a_{2i+1}, \ldots, a_{2n})(a_{2n+1})
+ (-1)^n \psi(a_1, \ldots, a_{2n-1})(a_{2n+1}),
\]
where $a_k \in \mathcal{A}^k (k = 1, 2, \ldots, 2n + 1)$, in particular, if $a_1 = \cdots = a_{2n+1} = e_{m_j}, (j = 1, 2, \ldots)$, then

$$
\phi(e_{m_1}, \ldots, e_{m_j}) = \psi(e_{m_1}, \ldots, e_{m_j}) = \frac{1}{\omega(m_j)^{2n+1} j^2}.
$$

So since $\omega(j) \leq 1/2^j$

$$
\|\psi\| \geq \sup_j \left\{ \left| \psi(\omega(m_j)e_{m_1}, \ldots, \omega(m_j)e_{m_j})(\omega(m_j)e_{m_j}) \right| \right\} \geq \sup_j \left\{ \frac{1}{\omega(m_j)^{2n+1} j^2} \right\} = \infty
$$

which is a contradiction. So $\mathcal{H}^{2n}(\mathcal{A}^k, (\mathcal{A}^k)') \neq 0$ and also $\mathcal{H}^{2n+1}(\mathcal{A}^k) \neq 0$.

3. Odd dimensional cohomology groups of weighted sequence algebras

In this section we will show that $\mathcal{H}^{2n+1}(\mathcal{A}^k, (\mathcal{A}^k)') \neq 0$ and also $\mathcal{H}^{2n+1}(\mathcal{A}^k) \neq 0$ for every $n \geq 1$. Note that the structure of the function $\phi$ which is a base for Theorem 3.4, for the three dimensional case is different from the structure of the corresponding functions in the other cases.

**Lemma 3.1.** Let $\sum_{i=-\infty}^{\infty} \alpha_i$ be an absolutely convergent series of real numbers, and let $\phi : \mathcal{A}^k \times \mathcal{A}^k \times \mathcal{A}^k \rightarrow (\mathcal{A}^k)'$ be the function defined by

$$
\phi(a, b, c)(d) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{a' \cdot b' \cdot c' \cdot d' - a' \cdot b' \cdot c' \cdot d'}{\omega(i)^2 \omega(j)^2} \alpha_i \alpha_j,
$$

where $a = a' + \alpha 1, b = b' + \beta 1, c = c' + \gamma 1$ and $d = d' + \lambda 1$. Then $\phi$ is a bounded cyclic $3$-cocycle.

**Proof.** It is easy to see that $\phi$ is a trilinear map and also

$$
|\phi(a, b, c)(d)| \leq \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{|a' \cdot b' \cdot c' \cdot d'|}{\omega(i)^2 \omega(j)^2} |\alpha_i||\alpha_j|
$$

$$
\leq 2\|a\|_{\omega^{-1}}\|b\|_{\omega^{-1}}\|c\|_{\omega^{-1}}\|d\|_{\omega^{-1}} \left( \sum_{i=-\infty}^{\infty} |\alpha_i| \right)^2.
$$
Thus $\phi$ is bounded and $\|\phi\| \leq 2\left\{ \sum_{n=-\infty}^{\infty} |\alpha_n| \right\}^2$. Now we want to show that $\phi$ satisfies

\[ a \alpha(b, c, d)(h) - \alpha(ab, c, d)(h) + \alpha(a, bc, d)(h) \]

\[-\phi(a, b, cd)(h) + (\phi(a, b, c)d)(h) = 0, \]

where $a = a' + \alpha1$, $b = b' + \beta1$, $c = c' + \gamma1$, $d = d' + \lambda1$ and $h = h' + \theta1$. By definition of $\phi$ and (1)

\[
\sum_i \sum_j \frac{\alpha_i \alpha_j}{\omega(i)^2 \omega(j)^2} \left( \left( b_{i}' c_{i}' d_{i}' h_{i}' a_{i}' + \alpha b_{i}' c_{i}' d_{i}' h_{i}' + \theta b_{i}' c_{i}' d_{i}' a_{i}' \right) \right.
\]

\[- \left. \left( b_{i}' c_{i}' d_{i}' h_{i}' a_{i}' + \alpha b_{i}' c_{i}' d_{i}' h_{i}' + \theta b_{i}' c_{i}' d_{i}' a_{i}' \right) \right) \right] = \sum_i \sum_j \frac{\theta b_{i}' c_{i}' d_{i}' a_{i}'}{\omega(i)^2 \omega(j)^2} \alpha_i \alpha_j - \sum_i \sum_j \frac{\theta a_{i}' b_{i}' c_{i}' d_{i}'}{\omega(i)^2 \omega(j)^2} \alpha_i \alpha_j = 0
\]

and so $\phi$ is a 3-cocycle. Also $\phi$ is cyclic, since

\[
\phi(d, a, b, c) = \sum_i \sum_j \frac{d_{i}' a_{i}' b_{i}' c_{i}' - d_{i}' a_{i}' b_{i}' c_{i}'}{\omega(i)^2 \omega(j)^2} \alpha_i \alpha_j
\]

\[ = - \sum_i \sum_j \frac{a_{i}' b_{i}' c_{i}' d_{i}'}{\omega(i)^2 \omega(j)^2} \alpha_i \alpha_j + \sum_i \sum_j \frac{a_{i}' b_{i}' c_{i}' d_{i}'}{\omega(i)^2 \omega(j)^2} \alpha_i \alpha_j
\]

\[ = - \phi(a, b, c, d) = (-1)^3 \phi(a, b, c, d). \]

Now we are going to construct the $2n - 1$-cocycle $\phi$ for higher dimensions.

**Lemma 3.2.** Let $\psi_{ij} : s^d \times s^d \times \cdots \times s^d \rightarrow (s^d)^{2n}$ be a $2n$-linear function defined by

\[
\psi_{ij}(a_1, \ldots, a_{2n})(a_{2n+1}) = \sum_{k=1}^{2n+1} a_{i_{1k}} \cdots a_{i_{j1}} \cdots a_{i_{2n+1k}},
\]
where \( a_k = a_k' + \beta_k 1 \) and \( a_k' = \{a_{ij}'\}_{i < j} \) \((k = 1, \ldots , 2n + 1)\). Then
\[
\delta \psi_{ij}(a_1, \ldots , a_{2n+1})(a_{2n+2}) = a_{ij}'a_{i+1}'+ \cdots a_{(2n+1)j}'a_{(2n+2)j}' + \sum_{k=1}^{2n+1} (-1)^k a_{ij}'a_{kj}'a_{(k+1)j}'a_{(2n+2)j}'.
\]

PROOF. By the coboundary formula we have
\[
(2) \quad \delta \psi_{ij}(a_1, \ldots , a_{2n+1})(a_{2n+2}) = \psi_{ij}(a_2, \ldots , a_{2n+1})(a_{2n+2}a_1)
+ \sum_{k=1}^{2n} (-1)^k \psi_{ij}(a_1, \ldots , a_ia_{k+1}, \ldots , a_{2n+1})(a_{2n+2})
- \psi_{ij}(a_1, \ldots , a_{2n})(a_{2n+1}a_{2n+2}).
\]

Using the definition of \( \psi_{ij} \) we obtain the value of all terms on the right-hand side of the above equation as follows
\[
\psi_{ij}(a_2, \ldots , a_{2n+1})(a_{2n+2}a_1)
= a_{ij}'a_{i+1}'a_{(2n+2)j}' + \sum_{k=2}^{2n+1} a_{ij}'a_{kj}'a_{(2n+2)j}'
+ \sum_{k=2}^{2n+2} \beta_1a_{ij}'a_{kj}'a_{(2n+2)j}'
+ \sum_{k=1}^{2n+1} \beta_2a_{i}a_{j}'a_{(2n+2)j}'
+ \sum_{k=1}^{2n+1} \beta_3a_{i}a_{j}'a_{(2n+2)j}'.
\]

For \( l = 1, \ldots , 2n, \)
\[
\psi_{ij}(a_1, \ldots , a_ia_{i+1}, \ldots , a_{2n+1})(a_{2n+2})
= a_{ij}'a_{i}a_{(i+1)}a_{i+1}'a_{(2n+2)j}' + \sum_{k=1}^{2n+2} a_{ij}'a_{kj}'a_{i}a_{(2n+2)j}'
+ \sum_{k=1}^{2n+2} a_{ij}'a_{kj}'a_{i}a_{(2n+2)j} + \sum_{k=1}^{2n+2} a_{ij}'a_{kj}'a_{i}a_{(2n+2)j} + \sum_{k=1}^{2n+2} a_{ij}'a_{kj}'a_{i}a_{(2n+2)j},
\]

where symbol \( \tilde{\ } \) shows the element in that position is removed.
\[
\psi_{ij}(a_1, \ldots , a_{2n})(a_{2n+1}a_{2n+2})
= a_{ij}'a_{i}a_{(2n+1)j}'a_{(2n+2)j}' + \sum_{k=1}^{2n} a_{ij}'a_{kj}'a_{i}a_{(2n+2)j}.
\]
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\[ + \sum_{k=1}^{2n+2} \beta_{2n+1} a_i' \cdots a_j' a_{(2n)} a_{(2n+2)} + \sum_{k=1}^{2n+1} \beta_{2n+2} a_i' \cdots a_j' a_{(2n+1)} \]

Substitute the values for \( \psi_{ij} \) obtained above in (2). Then all summations with \( \beta_k \) \((k = i, \ldots, 2n + 2)\) coefficients cancel in pairs, and we obtain

\[
\delta \psi_{ij}(a_1, \ldots, a_{2n+2}) \]

\[
= a_i a_j' \cdots a_{(2n+1)} a_{(2n+2)} + \sum_{k=1}^{2n+1} (-1)^k a_i(a_{(2n)} a_{(2n+2)}) \]

\[
+ \sum_{k=2}^{2n+1} a_i \cdots a_j' a_{(2n+2)} + \sum_{l=1}^{2n+2} (-1)^l \sum_{k=1}^{2n+1} a_i(a_{(2n)} a_{(2n+2)}) \]

and the sum of the last two terms is zero because, they contain 2n terms like \( a_i a_j' \cdots a_{(2n+2)} \) for every \( k = 1, \ldots, 2n + 2 \), half with a positive sign and the other half with a negative sign which cancel in pairs. So this finishes the proof.

**Lemma 3.3.** Let \( \sum_{i=1}^{2n+1} a_i \) be an absolutely convergent series of real numbers, and let \( \phi : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to (\mathbb{R}^n)' \) be the function defined by

\[
\phi(a_1, \ldots, a_{2n+1})(a_{2n+2}) = \sum_{j} \frac{a_i a_j}{\omega(i)2^n \omega(j)^2} \delta \psi_{ij}(a_1, \ldots, a_{2n+1})(a_{2n+2}),
\]

where \( \psi_{ij} \) is defined as in Lemma 3.2. Then \( \phi \) is a bounded cyclic \( (2n + 1) \)-cocycle for every \( n > 1 \).

**Proof.** It is easy to see that \( \phi \) is a 2n + 1-linear map and also

\[
|\phi(a_1, \ldots, a_{2n+1})(a_{2n+2})| \leq (2n + 2) \|a_1\| \cdots \|a_{2n+2}\|_{\omega^{-1}} \left( \sum \frac{|a_i|}{\omega(a_i)} \right)^2.
\]

Thus \( \phi \) is bounded and \( \|\phi\| \leq (2n + 2) \left( \sum_{i=1}^{\infty} |a_i| \right)^2 \). Also \( \phi \) is a \( (2n + 1) \)-cocycle, that is,

\[
\delta \phi = \sum_{i} \sum_{j} \frac{a_i a_j}{\omega(i)^2 \omega(j)^2} \delta \psi_{ij} = 0
\]

because \( \delta \psi_{ij} = 0 \). Furthermore we show that \( \phi \) is cyclic, that is, it satisfies

\[
\phi(a_1, \ldots, a_{2n+1})(a_{2n+2}) = (-1)^{2n+1} \phi(a_2, \ldots, a_{2n+2})(a_1).
\]
For this we have to calculate the right-hand side of the above equation. We have the following:

\[
\phi(a_1, \ldots, a_{2n+2})(a_i) = \sum_i \sum_j \left\{ \frac{\alpha_i \alpha_j}{\omega(i)^2 \omega(j)} \left( a_{ij} \cdots a_{(2n+2)i} a_{ij} - a_{ij} \cdots a_{(2n+2)i} a_{ij} \right) + a_{ij} \cdots a_{(2n+2)i} a_{ij} \right\}
\]

\[
= \sum_i \sum_j \left\{ \frac{\alpha_i \alpha_j}{\omega(i)^2 \omega(j)} \left( -a_{ij} \cdots a_{(2n+2)i} a_{ij} + a_{ij} \cdots a_{(2n+2)i} a_{ij} \right) + a_{ij} \cdots a_{(2n+2)i} a_{ij} \right\}
\]

\[
= -\phi(a_1, \ldots, a_{2n+1})(a_{2n+2}).
\]

Therefore \( \phi \) is a cyclic \( (2n + 1) \)-cocycle.

\[
\text{THEOREM 3.4. Let } \omega \text{ be a weight on } \mathbb{Z} \text{ such that } \inf\{\omega(i)\} = 0. \text{ Then }
\]

\[
\mathcal{H}^{2n+1}(\mathcal{O}^\omega, (\mathcal{O}^\omega)'') \neq 0
\]

and also \( \mathcal{H}^{2n+1}(\mathcal{O}^\omega) \neq 0 \) for every \( n \in \mathbb{N} \).

\[
\text{PROOF. Let } \phi \text{ be the bounded } 2n + 1 \text{-cocycle which was introduced in Lemma 3.1 for } n = 1 \text{ and in Lemma 3.3 for } n > 1. \text{ Consider the sequence } \alpha_i \text{ which was defined in the proof of Theorem 2.2. Note that } m_i \neq m_j \text{ whenever } i \neq j \text{ and } \omega(m_k) \leq 1/2^k.
\]

Also if \( i < j \), since \( 1/2^i < 1/2^j \), then \( \max\{\omega(m_i), \omega(m_j)\} \leq 1/2^i \).

Now if \( \psi \in \mathcal{O}^{2n}(\mathcal{O}^\omega, (\mathcal{O}^\omega)'') \) such that \( \phi = \delta \psi \), then by the definition of \( \phi \) and the coboundary formula we have

\[
\phi(e_{m_1}, e_{m_1}, \ldots, e_{m_1})(e_{m_1}) = \psi(e_{m_1}, e_{m_1}, \ldots, e_{m_1})(e_{m_1})
\]

\[
= \psi(e_{m_1}, e_{m_1}, \ldots, e_{m_1})(e_{m_1}) + \psi(e_{m_1}, e_{m_1}, \ldots, e_{m_1})(e_{m_1})
\]

\[
= \psi(e_{m_1}, e_{m_1}, e_{m_1}, e_{m_1})(e_{m_1}).
\]

Therefore by the definition of \( \phi \)

\[
\psi(e_{m_1} + e_{m_1}, e_{m_1}, \ldots, e_{m_1})(e_{m_1}) = \frac{\alpha_{m_1} \alpha_{m_1}}{\omega(m_1)^{2n} \omega(m_1)^2}.
\]
Suppose $\min \{ \omega(m_i), \omega(m_j) \} = C_{ij}$, then

$$\| C_{ij}(e_{m_i} + e_{m_j}) \|_{* - 1} = 1 \quad \text{and} \quad \| \omega(m_i)e_{m_i} \|_{* - 1} = 1.$$ 

If we let $i < j$, then

$$\| \psi \| \geq \sup_{i,j} \left\{ |\psi(C_{ij}(e_{m_i} + e_{m_j}), \omega(m_i)e_{m_i}, \ldots, \omega(m_i)e_{m_i})(\omega(m_j)e_{m_j})| \right\}$$

$$= \sup_{i,j} \left\{ \frac{\min \{ \omega(m_i), \omega(m_j) \} \alpha_{m_i} \alpha_{m_j}}{\omega(m_i)\omega(m_j)} \right\}$$

$$= \sup_{i,j} \left\{ \frac{1}{\max \{ \omega(m_i), \omega(m_j) \} i^2 j^2} \right\} \geq \sup_{i,j} \left\{ \frac{2^i}{j^2} \right\}.$$ 

In particular, for $j = i + 1$, we have $\| \psi \| \geq \sup_i 2^i / (i + 1)^4 = \infty$ which contradicts $\psi \in H^{2n}(\mathcal{A}^*, (\mathcal{A}^*)')$. So $H^{2n+1}(\mathcal{A}^*, (\mathcal{A}^*)') \neq 0$ and $H^{2n+1}(\mathcal{A}^*) \neq 0$. 

**Remark.** Consider the short exact sequence $0 \to \mathcal{A} \to \mathcal{L} \to \mathbb{C} \to 0$. The dual of this short exact sequence, is the short exact sequence,

$$0 \to \mathbb{C} \to (\mathcal{L}^*)' \to (\mathcal{L})' \to 0.$$ 

This gives the long exact sequence of cohomology (see [6, III. Corollary 4.11])

$$\cdots \to H^n(\mathcal{A}^*, \mathbb{C}) \to H^n(\mathcal{L}^*, (\mathcal{A}^*)') \to H^n(\mathcal{A}^*, \mathcal{L}^*) \to \cdots.$$ 

From this, one can show that $H^n(\mathcal{A}^*, \mathbb{C}) \neq 0$ for every $n \geq 2$.

As we noticed in Section 1, $E_N$ is an amenable closed subalgebra of $\mathcal{A}^*$. So $\mathcal{A}^*$ satisfies the conditions of [12, Theorem 2.6 and Theorem 5.1]. We can therefore apply Theorem 2.2 and Theorem 3.4 to conclude that for each $n \geq 2$, the $E_N$-relative (cyclic) cohomology of $\mathcal{A}^*$ does not vanish.

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**References**


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