INTERPOLATION PROBLEM FOR $\ell^1$ AND A UNIFORM ALGEBRA

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Abstract

Let $A$ be a uniform algebra and $M(A)$ the maximal ideal space of $A$. A sequence $\{a_n\}$ in $M(A)$ is called $\ell^1$-interpolating if for every sequence $\{a_n\}$ in $\ell^1$ there exists a function $f$ in $A$ such that $f(a_n) = a_n$ for all $n$. In this paper, an $\ell^1$-interpolating sequence is studied for an arbitrary uniform algebra. For some special uniform algebras, an $\ell^1$-interpolating sequence is equivalent to a familiar $\ell^\infty$-interpolating sequence. However, in general these two interpolating sequences may be different from each other.


Keywords and phrases: uniform algebra, $\ell^1$, interpolation, maximal ideal space, pseudo-hyperbolic distance.

1. Introduction

Let $A$ be a uniform algebra on a compact Hausdorff space $X$ and $M(A)$ the maximal ideal space of $A$. Throughout this paper we assume that $\{a_n\}$ is an infinite sequence of distinct points in $M(A)$. For $1 \leq p \leq \infty$, a sequence $\{a_n\}$ is called $\ell^p$-interpolating if for every sequence $\{a_n\}$ in $\ell^p$ there exists a function $f$ in $A$ such that $f(a_n) = a_n$ for all $n$.

For $A = H^\infty(D)$, the set of all bounded analytic functions on the unit disc $D$ in $\mathbb{C}$, an $\ell^\infty$-interpolating sequence was studied by Carleson [2] and Izuchi [4]. Carleson [2] determined an $\ell^\infty$-interpolating sequence when $\{a_n\}$ is in $D$, Izuchi [4] studied the general situation. Recently, Hatori [3] showed that an $\ell^1$-interpolating sequence is equivalent to an $\ell^\infty$-interpolating sequence when $\{a_n\}$ is in $D$. In this paper we study...
an $\ell^1$-interpolating sequence for an arbitrary uniform algebra $A$ when $\{a_n\}$ is in $M(A)$. For $\{a_n\}$ in $M(A)$ put

$$J = \{ f \in A; f = 0 \text{ on } \{a_n\}, \quad J_k = \{ f \in A; f = 0 \text{ on } \{a_n\}_{n \neq k} \}$$

and

$$\rho_k = \sup \{|f(a_k)|; f \in J_k, \|f\| \leq 1\}.$$ 

For $a, b$ in $M(A)$

$$\sigma(a, b) = \sup \{|f(a)|; f(b) = 0, \|f\| \leq 1\}.$$ 

When $A = H^\infty(D)$ and $\{a_n\}$ is in $D$,

$$\sigma(a_k, a_n) = \frac{|a_k - a_n|}{1 - \bar{a}_k a_n} \quad \text{and} \quad \rho_k = \prod_{n \neq k} \frac{|a_k - a_n|}{1 - \bar{a}_k a_n}.$$ 

In general, we do not know whether

$$\rho_k = \prod_{n \neq k} \sigma(a_k, a_n).$$ 

However, under some mild condition (Hypothesis I in Section 4), we can show that

$$\rho_k \geq \prod_{n \neq k} \sigma(a_k, a_n).$$ 

In general, $\rho_k > 0$ if and only if $J_k \supsetneq J$. Hence $\rho_k > 0$ if and only if there exists a function $f_k$ in $A$ such that $f_k(a_n) = \delta_{nk}$. In this paper, for $\{a_n\}$ in $M(A)$ we assume that $\rho_k > 0$ for all $k$.

In Section 2, for an arbitrary uniform algebra we show that $\{a_n\}$ is an $\ell^1$-interpolating sequence if and only if $\inf_k \rho_k > 0$. In Section 3, we define a finite $\ell^1$-interpolating sequence and give a necessary and sufficient condition to characterize it. In Section 4, we show that if $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$, then $\{a_n\}$ is always a finite $\ell^1$-interpolating sequence and under some mild condition it is an $\ell^1$-interpolating sequence. In some sense, this type of theorem for an $\ell^\infty$-interpolating sequence was conjectured in [1]. In Section 5, we apply the results from the previous sections to concrete uniform algebras. In Section 6, we comment on an $\ell^\infty$-interpolating sequence.

### 2. $\ell^1$-interpolating sequence

In this section we show that $\{a_n\}$ is an $\ell^1$-interpolating sequence if and only if $\inf_k \rho_k > 0$. The argument in the ‘only if’ part of Lemma 1 is similar to the one which was used by Hatori [3] when $A = H^\infty(D)$. 

Lemma 1. \(\{a_n\}\) is an \(\ell^1\)-interpolating sequence if and only if there exists a sequence \(\{f_n\}\) in \(A\) such that \(f_n(a_k) = \delta_{nk}\) \((n \geq 1, k \geq 1)\) and \(\sup_n \|f_n + J\| < \infty\).

Proof. Suppose \(M = \sup_n \|f_n + J\| < \infty\) and \(f_n(a_k) = \delta_{nk}\). Let \(\varepsilon\) be an arbitrary positive constant. For each \(n\) there exists \(g_n\) in \(J\) such that \(\|f_n + g_n\| \leq M + \varepsilon\). If \((a_n) \in \ell^1\), put

\[
 f = \sum_{n=1}^{\infty} \alpha_n (f_n + g_n).
\]

Then \(f\) belongs to \(A\) and \(f(a_n) = \alpha_n\) for \(n = 1, 2, \ldots\). Suppose \(S = \{a_n\}\) is an \(\ell^1\)-interpolating sequence. Then there exists a sequence \(\{f_n\}\) in \(A\) such that \(f_n(a_k) = \delta_{nk}\). For \((a_n) \in \ell^1\), put

\[
 T(a_n) = \sum_{n=1}^{\infty} \alpha_n f_n|S,
\]

then by hypothesis there exists a function \(f\) such that \(T(a_n) = f|S\). Since \(A|S\) is algebraically isomorphic to the quotient algebra \(A/J\), we use the quotient norm of \(A/J\) in \(A|S\). By the closed graph theorem, \(T\) is bounded from \(\ell^1\) to \(A|S\) and so

\[
 \|f_k\| = \|f_k|S\| \leq \|T\|
\]

because \(T(\{\delta_{nk}\}) = f_k|S\).

Lemma 2. Suppose \(\{f_n\}\) is a sequence in \(A\) such that \(f_n(a_k) = \delta_{nk}\). Then

\[
 \|f_n + J\| = 1/\rho_n \quad \text{for} \quad n = 1, 2, \ldots.
\]

Proof. Since \((\rho_n f_n)(a_k) = \rho_n \delta_{nk}\), \(\|\rho_n f_n + J\| \geq 1\). By definition of \(\rho_n\), for each \(l \geq 1\) there exists \(g_l \in A\) such that \(\|g_l\| = 1\), \(g_l(a_n) = 0\) for \(n \neq k\) and

\[
 \rho_k - 1/l \leq g_l(a_k) \leq \rho_k.
\]

Put \(G_l = g_l/g_l(a_k)\), then \(G_l \in A\) and

\[
 \frac{1}{\rho_k} \leq \|G_l\| = \frac{1}{|g_l(a_k)|} \leq \frac{1}{\rho_k - 1/l}.
\]

Moreover, \(G_l(a_k) = 1\), \(G_l(a_n) = 0\) for \(n \neq k\) and so \(G_l \in f_k + J\). Since \(\|f_k + J\| \leq (\rho_k - 1/l)^{-1}\) for any \(l \geq 1\), \(\|\rho_k f_k + J\| \leq 1\).

Theorem 1. Let \(A\) be an arbitrary uniform algebra and let \(\{a_n\}\) be in \(M(A)\). Then \(\{a_n\}\) is a \(\ell^1\)-interpolating sequence if and only if \(\inf_k \rho_k > 0\).

Proof. The proof follows from Lemma 1 and Lemma 2.
3. Finite $\ell^1$-interpolating sequence

We say that $\{a_n\}$ is a finite $\ell^1$-interpolating sequence if there exists a finite positive constant $\gamma$ which satisfies the following: For any finite $l \geq 1$ and for any $(a_n)$ in the unit ball of $\ell^1$, there exists a function $F_l$ in $A$ such that

$$F_l(a_n) = a_n \quad \text{for } 1 \leq n \leq l$$

and $\|F_l\| \leq \gamma$.

For $\{a_n\}$ in $M(A)$ and $1 \leq k \leq l < \infty$, put

$$J^i = \{ f \in A; \ f(a_n) = 0 \text{ if } 1 \leq n \leq l \},$$

$$J'_k = \{ f \in A; \ f(a_n) = 0 \text{ if } 1 \leq n \leq l, \ n \neq k \}$$

and

$$\rho_{k,l} = \sup \{ |f(a_n)|; \ f \in J'_k, \ |f| \leq 1 \}.$$  

Then $\rho_{k,l} \geq \rho_{k,l+1}$ and $\lim_{l \to \infty} \rho_{k,l} \geq \rho_k$.

**Lemma 3.** $\{a_n\}$ is a finite $\ell^1$-interpolating sequence if and only if for each $l \geq 1$ there exists a sequence $(f_{i,n})_{n=1}^l$ in $A$ such that $f_{i,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq L$ and $\sup_i \sup_{1 \leq n \leq l} \|f_{i,n} + J^i\| < \infty$.

**Proof.** $(a_n)$ denotes an element in the unit ball of $\ell^1$. Suppose

$$M = \sup_i \sup_{1 \leq n \leq l} \|f_{i,n} + J^i\| < \infty$$

and $f_{i,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq l$, then for any finite $l \geq 1$

$$\left\| \sum_{n=1}^l a_n f_{i,n} + J^i \right\| \leq \left( \sum_{n=1}^l |a_n| \right) M.$$

If $\gamma = M + 1$, then for any $l \geq 1$ there exists $g_l \in J^i$ such that $\left\| \sum_{n=1}^l a_n f_{i,n} + g_l \right\| \leq \gamma$. Set $F_l = \sum_{n=1}^l a_n f_{i,n} + g_l$, then $F_l(a_n) = a_n$ for $1 \leq n \leq l$ and $\|F_l\| \leq \gamma$. Suppose $\{a_n\}$ is a finite $\ell^1$-interpolating sequence. Since $\{a_n\}$ is an infinite sequence of distinct points in $M(A)$, for each $l \geq 1$ there exists a sequence $(f_{i,n})_{n=1}^l$ in $A$ such that $f_{i,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq n$. Put

$$T^i(a_n) = \sum_{n=1}^l a_n f_{i,n} + J^i;$$

then $\left\| T^i(a_n) \right\| \leq \left\| T^i \left( \sum_{n=1}^l |a_n| \right) \right\|$. If $\left\| T^i \right\| \to \infty$ as $l \to \infty$, then there exists $(a_n)$ in the unit ball of $\ell^1$ such that $\left\| T^i(a_n) \right\| \to \infty$ as $l \to \infty$. On the other hand, by hypothesis $\left\| T^i(a_n) \right\| \leq \gamma < \infty$ for all $l$. This contradiction implies that $M = \sup_l \|T^i\| < \infty$. This shows that for any $l \geq 1$ and any $k \geq 1$ with $k \leq l$,

$$\|f_{i,k} + J^i\| = \|T^i(\delta_{kn})\| \leq M.$$  

$\square$
LEMMA 4. For \( l = 1, 2, \ldots \) and \( 1 \leq k \leq l \), \( \| f_k + J^l \| = 1/\rho_{k,l} \).

Proof is almost the same as the proof of Lemma 2.

THEOREM 2. Let \( A \) be an arbitrary uniform algebra and let \( \{ a_n \} \) be in \( M(A) \). Then \( \{ a_n \} \) is a finite \( \ell^1 \)-interpolating sequence if and only if \( \inf_k \lim_{l \to \infty} \rho_{k,l} > 0 \).

PROOF. The statement of the theorem follows from Lemma 3 and Lemma 4.

4. Uniformly separated sequence

When \( A = H^\infty(D) \) and \( \{ a_n \} \) is in \( D \), for any \( k \geq 1 \)

\[
\rho_k = \prod_{n \neq k} \sigma(a_k, a_n) = \lim_{l \to \infty} \rho_{k,l}.
\]

When \( \{ a_n \} \) is in \( M(A) \), Izuchi [4] showed essentially that \( \inf_k \rho_k > 0 \) implies \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \). However, this is not true in general. If \( \sum_{n=1}^{\infty}(1 - \rho_n) < \infty \), then \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \). In fact, \( \rho_n \leq \sigma(a_k, a_n) \) for \( n \neq k \) and so \( \prod_{n=1}^{\infty} \rho_n \leq \prod_{n \neq k} \sigma(a_k, a_n) \) for any \( k \geq 1 \). When \( \sum_{n=1}^{\infty}(1 - \rho_n) < \infty \), \( 0 < \prod_{n=1}^{\infty} \rho_n \) and so \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \). In this section, we study these three quantities.

LEMMA 5. (1) For any \( l \geq 1 \), \( \rho_{k,l} \geq \prod_{n \neq k} \sigma(a_k, a_n) \). Hence for any \( k \geq 1 \)

\[
\lim_{l \to \infty} \rho_{k,l} \geq \prod_{n \neq k} \sigma(a_k, a_n).
\]

(2) For \( 1 \leq n \leq l \) and \( n \neq k \), \( \rho_{k,l} \leq \sigma(a_k, a_n) \). Hence for any \( k \geq 1 \)

\[
\lim_{l \to \infty} \rho_{k,l} \leq \inf_{n \neq k} \sigma(a_k, a_n).
\]

PROOF. (1) Fix any positive constant \( \varepsilon > 0 \). For each \( n \) with \( l \geq n \geq 1 \) and \( n \neq k \), there exists \( F_n^x \in A \) such that \( \| F_n^x \| \leq 1 \), \( F_n^x(a_n) = 0 \) and \( \sigma(a_k, a_n) \geq |F_n^x(a_k)| \geq \sigma(a_k, a_n) - \varepsilon \). Then \( F^x = \prod_{n \neq k} F_n^x \) belongs to \( J_{l,k} \), \( \| F^x \| \leq 1 \) and

\[
\rho_{l,k} \geq |F^x(a_k)| \geq \prod_{n \neq k} [\sigma(a_k, a_n) - \varepsilon].
\]

As \( \varepsilon \to 0 \), \( \rho_{l,k} \geq \prod_{n \neq k} \sigma(a_k, a_n) \) for any \( l \geq 1 \) and hence

\[
\lim_{l \to \infty} \rho_{k,l} \geq \prod_{n \neq k} \sigma(a_k, a_n).
\]

(2) is clear by the definitions of \( \rho_{k,l} \) and \( \sigma(a_k, a_n) \) for \( 1 \leq n \leq l \) and \( n \neq k \).
THEOREM 3. Let $A$ be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$.

(1) If $\inf_k \prod_{a \in A} \sigma(a_k, a_n) > 0$, then $\{a_n\}$ is a finite $\ell^1$-interpolating sequence.

(2) If $\{a_n\}$ is a finite $\ell^1$-interpolating sequence, then $\inf_{n \neq k} \sigma(a_k, a_n) > 0$.

**Proof.** (1) By Lemma 5 (1), $\inf_k \lim_{n \to \infty} \rho_{k,l} > 0$ and so, by Theorem 2, $\{a_n\}$ is a finite $\ell^1$-interpolating sequence.

(2) By Theorem 2 $\inf_k \lim_{n \to \infty} \rho_{k,l} > 0$ and so, by Lemma 5 (2), $\inf_{n \neq k} \sigma(a_k, a_n) > 0$. $\Box$

**Hypothesis I.** Let $A$ be a uniform algebra and let $\{a_n\}$ be in $M(A)$. If $g_l$ is a function in $A$ and $\|g_l\| \leq 1$ for $l = 1, 2, \ldots$, then there exist a subsequence $\{g(l_j)\}$ of $\{g_l\}$ and a function $g$ in $A$ such that $\|g\| \leq 1$ and $\lim_{j \to \infty} g(l_j)(a_n) = g(a_n)$ for any $n \geq 1$.

**Hypothesis II.** Let $A$ be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. For any $a, b$ in $\{a_n\}$ with $a \neq b$, if the function $f$ in $A$ satisfies $f(a) = f(b) = 0$ and $\|f\| \leq 1$, then for any $\epsilon > 0$ there exist two functions $g$ and $h$ in $A$ such that $\|g\| \leq 1 + \epsilon, \|h\| \leq 1 + \epsilon, g(a) = 0, h(b) = 0$ and $f = gh$.

**Lemma 6.** Let $A$ be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. If $\{a_n\}$ satisfies Hypothesis I, then $\lim_{n \to \infty} \rho_{k,l} = \rho_k$ for any $k \geq 1$, and hence a finite $\ell^1$-interpolating sequence is an $\ell^1$-interpolating sequence.

**Proof.** $\lim_{n \to \infty} \rho_{k,l} \geq \rho_k$ for any $k \geq 1$. If $\lim_{n \to \infty} \rho_{k,l} > \epsilon > 0$, then for each $l$ there exists $g_l \in J_k^l$ such that $\|g_l\| \leq 1$ and $|g_l(a_k)| \geq \epsilon > 0$. By hypothesis, there exists $g \in J_k$ such that $\|g\| \leq 1$ and $|g(a_k)| \geq \epsilon > 0$. Thus $\lim_{n \to \infty} \rho_{k,l} \leq \rho_k$ and so $\lim_{n \to \infty} \rho_{k,l} = \rho_k$. This together with Theorem 1 and Theorem 2 also imply that a finite $\ell^1$-interpolating sequence is an $\ell^1$-interpolating sequence. $\Box$

**Lemma 7.** Assume Hypothesis II. If $f$ is a function in $J_{k-1}$ with $\|f\| \leq 1$, then for any $\epsilon > 0, f = \prod_{a \in A} f(a) = 0$ (with $a$ and $\|f\| \leq (1 + \epsilon)^{-1}$.

**Proof.** We may assume $k = 1$. Fix any $\epsilon > 0$. By Hypothesis II, $f = g_2 g_3, \|g_j\| \leq 1 + \epsilon (j = 2, 3)$ and $g_2(a_2) = g_3(a_3) = 0$. Since $f(a_1) = 0, g_2(a_1) = 0$ or $g_3(a_1) = 0$. We may assume $g_2(a_1) = 0$. By Hypothesis II, $g_2 = g_2 g_3, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10} \leq (1 + \epsilon)^2 (j = 2, 4), h_2(a_2) = h_4(a_4) = 0$. Hence there exist $h_2, h_3, h_4$ such that $f = h_2 h_3 h_4, \|h_j\| \leq (1 + \epsilon)^2 (j = 2, 3, 4) h_2(a_2) = h_3(a_3) = h_4(a_4) = 0$. This argument implies the proof. $\Box$

**Lemma 8.** Let $A$ be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. If $\{a_n\}$ satisfies Hypothesis II, then for $1 \leq k \leq l$, $\rho_{k,l} = \prod_{a \in A} \sigma(a_k, a_n)$. Moreover, if $\{a_n\}$ satisfies Hypothesis I, then $\rho_k = \prod_{a \in A} \sigma(a_k, a_n)$. 

PROOF. By (1) of Lemma 5 it is sufficient to show that \( \rho_{k,l} \leq \prod_{n \neq k}^{l} \sigma(a_k, a_n) \). If \( 0 < \delta < \rho_{k,l} \), then there exists \( f \in J_{k,l} \) with \( \|f\| \leq 1 \) such that

\[
\rho_{k,l} - \delta \leq |f(a_k)| \leq \rho_{k,l}.
\]

For any \( \varepsilon > 0 \), by Lemma 7, \( f \) can be factorized as \( f = \prod_{n \neq k}^{l} f_n \), \( \|f_n\| \leq (1 + \varepsilon)^{l-1} \) and \( f_n(a_n) = 0 \) for \( n \neq k \). Hence

\[
\prod_{n \neq k}^{l} |f_n(a_k)| \leq (1 + \varepsilon)^{(l-1)(l-1)} \prod_{n \neq k}^{l} \sigma(a_k, a_n).
\]

As \( \varepsilon \to 0 \), \( \rho_{k,l} - \delta \leq \prod_{n \neq k}^{l} \sigma(a_k, a_n) \). Since \( \delta \) is arbitrary, \( \rho_{k,l} \leq \prod_{n \neq k}^{l} \sigma(a_k, a_n) \).

\[ \square \]

THEOREM 4. Let \( A \) be an arbitrary uniform algebra and let \( \{a_n\} \) be in \( M(A) \).

(1) Under Hypothesis II, \( \{a_n\} \) is a finite \( \ell^1 \)-interpolating sequence if and only if \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \).

(2) Under Hypotheses I and II, \( \{a_n\} \) is an \( \ell^1 \)-interpolating sequence if and only if \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \).

PROOF. Theorem 1, Theorem 2 and Lemma 8 imply the theorem. \[ \square \]

When \( A = H^\infty(D) \) and \( \{a_n\} \) is in \( D \), \( \{a_n\} \) satisfies Hypotheses I and II. Let \( A \) be a disc algebra. Then if \( \{a_n\} \) is in \( D \), then \( \{a_n\} \) satisfies Hypothesis II (see Section 5). On the other hand, it is easy to see that there exists a sequence \( \{a_n\} \) in \( D \) which does not satisfy Hypothesis I.

5. Special uniform algebras

When \( A = H^\infty(D) \) and \( \{a_n\} \) is in \( D \), Hatori [3] showed that \( \{a_n\} \) is an \( \ell^1 \)-interpolating sequence if and only if \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \). Since it is clear that \( \{a_n\} \) in \( D \) satisfies Hypotheses I and II, this is a corollary of (2) of Theorem 4. Corollary 3 is also a result of Hatori [3]. We give another proof of it. Hatori [3] also shows this type of theorem for a Hardy space \( H^p \) (\( 1 \leq p < \infty \)) on a finite open Riemann surface and generalizes a theorem of Shapiro and Shields [7].

COROLLARY 1. Let \( A \) be a uniform closed algebra between the disc algebra \( \mathcal{S} \) and \( H^\infty(D) \), and let \( \{a_n\} \) be in \( D \). Suppose that \( f/z \) belongs to \( A \) for \( f \) in \( A \) with \( f(0) = 0 \). Then \( \{a_n\} \) is a finite \( \ell^1 \)-interpolating sequence if and only if \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \).
PROOF. If \( f \in A \) and \( f(a) = 0 \) for some \( a \in D \), then \( f(z)/(z-a) \) belongs to \( A \) (see [5]). Hence

\[
\frac{1 - \tilde{a}z}{z-a} f(z) \quad \text{belongs to} \quad A
\]

and \((z-a)/(1 - \tilde{a}z)\) is a unimodular function in \( \mathcal{S} \). Therefore, \( \{a_n\} \) satisfies Hypothesis II and so (1) of Theorem 4 implies the corollary. \( \square \)

**Corollary 2.** Let \( A = H^\infty(D^m) \) and let \( \{a_n\} \) be in \( D^m \). Suppose \( a_n = (a_n^1, a_n^2, \ldots, a_n^n) \) and \( \sum_{n=1}^{\infty} (1 - |a_n^l|) < \infty \) for \( 1 \leq l \leq m \). Then \( \{a_n\} \) is an \( \ell^1 \)-interpolating sequence if and only if \( \inf \prod_{n \neq k} \sigma(a_k, a_n) > 0 \).

**Proof.** By Theorem 2 and Lemma 6, the ‘if’ part is proved. We will prove the ‘only if’ part. Let

\[
B_k = B_k(z_1, \ldots, z_m) = \prod_{l=1}^{m} \prod_{n \neq k} \frac{-a_n^l z_l - a'_n}{|a_n^l| - \tilde{a}_n^l z_l},
\]

then \( B_k \) belongs to \( H^\infty(D^m) \) because \( \sum_{n=1}^{\infty} (1 - |a_n^l|) < \infty \) for \( 1 \leq l \leq m \). If \( F_k = B_k/B_k(a_k) \), then \( F_k(a_k) = \delta_{nk} \) and

\[
\|F_k + J\| = \|B_k(a_k)|^{-1}\|B_k + J\| = |B_k(a_k)|^{-1};
\]

thus \( \rho_k = |B_k(a_k)| \). Theorem 1 implies that \( \inf_k |B_k(a_k)| = \inf_k \rho_k > 0 \). Since

\[
\sigma(a_k, a_n) = \max \left( \left| \frac{a_n^1 - a_k^1}{1 - \tilde{a}_n a_k^1} \right|, \ldots, \left| \frac{a_n^n - a_k^n}{1 - \tilde{a}_n a_k^n} \right| \right)
\]

(see [1, page 162]),

\[
|B_k(a_k)| \leq \prod_{k \neq n} \sigma(a_k, a_n).
\]

This proves the corollary. \( \square \)

**Corollary 3.** Let \( R \) be a finite open Riemann surface and \( A = H^\infty(R) \) the set of all bounded analytic functions on \( R \). Then \( \{a_n\} \) in \( R \) is an \( \ell^1 \)-interpolating sequence if and only if \( \inf \prod_{n \neq k} \sigma(a_k, a_n) > 0 \).

**Proof.** It is known [8] that \( \{a_n\} \) is an \( \ell^\infty \)-interpolating sequence if and only if \( \inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0 \). If \( \{a_n\} \) is an \( \ell^1 \)-interpolating sequence, then \( \inf_k \rho_k > 0 \) by Theorem 1 and so by [8, Theorem 5.9] \( \{a_n\} \) is a \( \ell^\infty \)-interpolating sequence. \( \square \)

Let \( D_n = \{z \in \mathbb{C} : |z - c_n| < r_n\}, c_n > 0 \) as \( D_n \cap D_m = \emptyset \) (\( n \neq m \)), \( D_n \subset D \setminus \{0\} \) (\( n = 1, 2, 3, \ldots \)) and \( \sum_{n=1}^{\infty} r_n/c_n < \infty \). \( U = D \setminus \bigcup_{n} D_n \) is called a Zalcman domain [9]. When \( A = H^\infty(U) \) and \( \{a_n\} \) is in \( U \), if \( \inf \prod_{n \neq k} \sigma(a_k, a_n) > 0 \), then \( \{a_n\} \) is an \( \ell^1 \)-interpolating sequence by (1) of Theorem 3 and Lemma 6 because \( \{a_n\} \) satisfies Hypothesis I but \( \{a_n\} \) is not necessarily an \( \ell^\infty \)-interpolating sequence by [6].
6. $\ell^\infty$-interpolating sequence

When $\{a_n\}$ in $M(A)$ satisfies Hypothesis I, it is interesting to give a sufficient condition or a necessary condition for an $\ell^\infty$-interpolating sequence. Berndtsson, Chang and Lin [1] give the following problem: Let $A = H^\infty(Y)$ and let $\{a_n\} \subset Y$ be a bounded domain $Y \subset \mathbb{C}^n$. Suppose $\inf k \prod_{n \neq k} \sigma(a_k, a_n) > 0$. Is $\{a_n\}$ an $\ell^\infty$-interpolating sequence? In Proposition 1, $\sum_{n=1}^{\infty} (1 - \rho_n) < \infty$ and so by the remark above Lemma 5, $\inf k \prod_{n \neq k} \sigma(a_k, a_n) > 0$.

**Proposition 1.** Let $A$ be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. Suppose $\{a_n\}$ satisfies Hypothesis I. If $\rho_n \geq 2(n')^2/[(n' + 1)^2 + (n' + 2)^2]$ for $n = 1, 2, 3, \ldots$ and some $t > 1$, then $\{a_n\}$ is an $\ell^\infty$-interpolating sequence.

**Proof:** By Hypothesis I there exists a sequence $\{F_n\}$ in $A$ such that $\|F_n\| \leq 1$, $F_n(a_k) = 0$ if $k \neq n$ and $|F_n(a_n)| = \rho_n$ for $n = 1, 2, \ldots$. Izuchi [4, Theorem 1] has essentially proved the theorem. We use the notation from [4, Theorem 1]. Set

$$\delta_n = \frac{1}{1 + 2\rho_n}.$$

By the proof of [4, Theorem 1], there exists a sequence $G_n \in A$ such that

$$\sum_{n=1}^{\infty} |G_n| \leq \sum_{n=1}^{\infty} (1 + \epsilon_n) < \infty \text{ on } X.$$

Hypothesis I implies that $\{a_n\}$ is an $\ell^\infty$-interpolating sequence.

**Proposition 2.** Let $A$ be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. Suppose $\{f_k\}$ is a sequence in $A$ such that $f_k(a_n) = \delta_{nk}$. Then $\{a_n\}$ is an $\ell^p$-interpolating sequence if and only if

$$\sup_{\phi \in A^* \cap J^\perp} \left( \sum_{n=1}^{\infty} |\phi(f_n)|^p \right)^{1/p} < \infty,$$

where $1/p + 1/q = 1$ and $A^* \cap J^\perp = \{\phi \in A^*; \phi = 0 \text{ on } J\}$. For $p = 1$ and $q = \infty$ we assume that

$$\sup_{\phi} \left( \sum_{n=1}^{\infty} |\phi(f_n)|^q \right)^{1/q} = \sup_n |\phi(f_n)| = \sup_n \|f_n + J\|.$$
Proof. Suppose that
\[ \sup_{\phi \in A^* \cap J^* \setminus \{ \phi \}} \left( \sum_{n=1}^{\infty} |\phi(f_n)|^q \right)^{1/q} = \gamma_q < \infty. \]
For any \( \phi \in A^* \cap J^* \) with \( \| \phi \| \leq 1 \) and any \( l < \infty \),
\[ \left| \phi \left( \sum_{n=1}^{l} \alpha_n f_n \right) \right| \leq \left( \sum_{n=1}^{l} |\alpha_n|^\alpha \right)^{1/p} \left( \sum_{n=1}^{l} |\phi(f_n)|^q \right)^{1/q} \]
and so
\[ \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\| \leq \gamma_q \left( \sum_{n=1}^{\infty} |\alpha_n|^p \right)^{1/p} \],
where \( \tilde{f}_n = f_n + J \). Thus if \( \{\alpha_n\} \in \ell^p \) then \( \tilde{f} = \sum_{n=1}^{\infty} \alpha_n \tilde{f}_n \) belongs to \( A/J \). Then \( f(\alpha_n) = \alpha_n \) for \( n = 1, 2, \ldots \) and so \( \{\alpha_n\} \) is an \( \ell^p \)-interpolating sequence. Conversely, suppose \( S = \{\alpha_n\} \) is an \( \ell^p \)-interpolating sequence. For \( (\alpha_n) \in \ell^p \), set
\[ T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n f_n |S| \]
then there exists a function \( f \) such that \( T(\alpha_n) = f |S| \). Since \( T \) turns out to be bounded from \( \ell^1 \) to \( A/J \) (see Lemma 1), for \( \phi \in A^* \cap J^* \) with \( \| \phi \| \leq 1 \) we have
\[ |\phi(f)| = \left\| \sum_{n=1}^{\infty} \alpha_n \phi(f_n) \right\| \leq \| T \left( \sum_{n=1}^{\infty} |\alpha_n|^p \right)^{1/p} \).
Hence \( \sup_{\phi \in A^* \cap J^* \setminus \{ \phi \}} \left( \sum_{n=1}^{\infty} |\phi(f_n)|^q \right)^{1/q} < \infty. \)

Hatori [3] is interested in when an \( \ell^1 \)-interpolating sequence is an \( \ell^\infty \)-interpolating sequence. He showed that if \( A = H^\infty(R) \) and \( \{a_n\} \) in \( R \), then \( \{a_n\} \) is such a sequence (see Corollary 3). In general, Proposition 2 gives a necessary and sufficient condition for this to happen.

References


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