Three hypergeometric weeks at MATRIX
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Ling Long*, Masha Vlasenko and Wadim Zudilin †**

For some of us, the beginning of 2017 has been a tremendous mathematical experience at the MATRIX Institute in Creswick — a new research initiative of the University of Melbourne to host international programs on maths related topics in a naturally isolated and beautiful part of Victoria. The acronym MATRIX denotes the international MAThematical Research Institute by the CruX (the constellation commonly known as the Southern Cross), and there were indeed numerous nights when the participants of the three-week January program on *Hypergeometric motives and Calabi–Yau differential equations* could watch the starry sky with the naked eye. But those moments were short breaks in an intense series of lectures, discussions and collaboration.

To say more about the principal activity during the period 8–28 January 2017, we must first mention some background and recent developments of the theme of the program. An ultimate goal of arithmetic algebraic geometry is 'understanding' (that is, computing and relating to other objects) L-functions of algebraic varieties. By the influential Langlands program, these L-functions are expected to coincide with L-functions arising from automorphic forms, and consequently satisfy functional equations and analytic continuation; in other words, to behave like the classical Riemann zeta function. Recent work led by F. Rodriguez Villegas in collaboration with F. Beukers, H. Cohen, A. Mellit, D. Roberts, M. Watkins and others is targeted at investigating and computing explicitly the L-functions of the so-called hypergeometric motives. These are families of motives characterised by the fact that their periods are given by generalised hypergeometric functions. They form an important class of special functions, playing a crucial role in parts of physics such as conformal field theory and quantum mechanics. The L-functions of the hypergeometric motives are expected to cover a wide range (if not all) of known L-functions. The hypergeometric data allow us to efficiently compute parameters — degrees, Hodge numbers etc. — associated with the (sometimes unknown) algebraic structures. These data also provide us with a way to test conjectures on special values and the distribution of zeroes, while simultaneously verifying numerically standard conjectures on analytic continuation and functional equations for these L-functions. Many algorithms for computing the hypergeometric motives are now implemented in the mathematical software Magma.

Another step towards understanding the local factors of such L-functions was brought to the scene in recent works of R. Barman, F. Beukers, H. Cohen, J. Fuselier, S. Frechette, L. Long, D. McCarthy, A. Mellit, R. Osburn, R. Ramakrishna,

*Louisiana State University, Baton Rouge, LA, USA.
**Institute of Mathematics of the Polish Academy of Sciences, Warsaw, Poland.
†The University of Newcastle, Callaghan, NSW, Australia.

A. Straub, H. Swisher, F.-T. Tu and others by investigating finite hypergeometric functions and the analogies between them and classical hypergeometric functions. The techniques developed by the authors allow us to efficiently transport classical formulas to the finite field settings, count points on algebraic varieties over finite fields, study their congruence properties and Galois representations. More importantly, these new methods give us a way to interpret finite hypergeometric functions as periods of varieties over finite fields. This opens the door to the study of finite field analogues of multivariable hypergeometric functions, which are expected to correspond to more general motives, including multi-parameter families of Calabi–Yau manifolds.

An overall expectation is that the periods over finite fields form a new angle of understanding the integrality phenomenon arising in mirror symmetry. Originally discovered by physicists in the mid-1980s, mirror symmetry remains one of the central research themes binding string theory and algebraic geometry. Numerous examples show that the expression of the mirror map in so-called canonical coordinates possesses rich arithmetic properties, such as modularity. This expression involves particular solutions to a Picard–Fuchs differential equation of a family of Calabi–Yau manifolds near a singular point. Such equations are known as Calabi–Yau differential equations, and form a subject of intensive study during the last two decades. The contributors in gaining the knowledge about the equations and their arithmetic significance are from both physics and mathematics; the list of names include G. Almkvist, D. Broadhurst, É. Delaygue, A. Salerno, D. van Straten, J. Voight, Y. Yang, W. Zudilin and others. Quite remarkably, Calabi–Yau differential equations show up in several other contexts as diverse as rational approximations to π, Mahler measures and generating functions of random walks in models of statistical mechanics.

The focus of the MATRIX program was on developing new methods in investigating hypergeometric motives, on establishing the modularity of L-functions of Calabi–Yau manifolds and analysing the integrality phenomenon in mirror symmetry from the arithmetic point of view, as well as on various applications of finite hypergeometric functions and Calabi–Yau differential equations. The activity was an exceptional opportunity to unite international experts in two related areas of number theory and to introduce the Australian mathematicians with expertise in neighbouring topics into the play.

Week 1 was dedicated to hypergeometric motives and finite hypergeometric functions, and comprised of the talks by D. Broadhurst, R. Ramakrishna and W. Zudilin as well as mini-courses ‘Hypergeometric motives’ run by F. Rodriguez Villegas, D. Roberts and M. Watkins, and ‘Finite hypergeometric sums and Dwork cohomology’ by F. Beukers. The workshop of Week 2 on hypergeometric motives and Calabi–Yau differential equations was designed to draw a strong connection between the two topics and unite the international experts working on different

aspects of them. The talks delivered during the workshop were as follows:

Monday 16 January 2017

- Fernando Rodriguez Villegas: 'p-adic hypergeometric sums'
  David Broadhurst: 'Walks at sunrise with Gauss, Bessel, Kloosterman, Calabi and Yau'
- Abdellah Sebbar: 'Schwarzian differential equations and equivariant functions'
- Yifan Yang: 'Special values of hypergeometric functions and periods of CM elliptic curves'
- Ravi Ramakrishna: 'Some supercongruences for truncated hypergeometric series'

Tuesday 17 January 2017

- Adriana Salerno: 'Arithmetic mirror symmetry of $K3$ surfaces and hypergeometric functions'
- David Roberts: 'Hypergeometric motives and an unusual application of the Guinand–Weil–Mestre explicit formula'
- James Wan: 'Ramanujan-type series for $1/\pi$'
- Fang-Ting Tu: 'Supercongruences occurred to rigid hypergeometric type Calabi–Yau threefolds'

Wednesday 18 January 2017

- Frits Beukers: 'Explicit hypergeometric motives'
- Robert Osburn: 'Sequences, modular forms and cellular integrals'

Thursday 19 January 2017

- John Voight: 'Triangular modular curves'
- Holly Swisher: 'Hypergeometric functions over finite fields'
- Mark Watkins: 'Jacobi sum motives and Grossencharacters'
- Sharon Frechette: 'Finite-field Appell–Lauricella hypergeometric functions'
- Piotr Achinger: 'Canonical liftings of ordinary Calabi–Yau varieties'

Friday 20 January 2017

- Éric Delaygue: 'Arithmetic properties of hypergeometric mirror maps and Dwork congruences'
- Duco van Straten: 'Calabi–Yau equations and L-functions'
- Masha Vlasenko: 'On p-adic unit root formulas'

There was a special lecture by F. Rodriguez Villegas scheduled on late Thursday at the very last moment, in which he reported on his joint work 'Motivic supercongruences' with D. Roberts and M. Watkins, that had come out entirely in Creswick from discussions and talks during the program and workshop. This lecture highly influenced the design of final Week 3, which was aimed at investigating arithmetic and combinatorial aspects of periods of Calabi–Yau manifolds, in particular, the differential equations satisfied by them as well as the modularity aspects and numerous applications, and also on the latest development of Dwork's p-adic machinery. The last week featured by the mini-courses 'Calabi–Yau differential equations' by D. van Straten and 'Arithmetic properties of Calabi–Yau

equations and Dwork's congruences' by É. Delaygue and seminar talks by R. Osburn, A. Salerno, A. Sebbar, M. Vlasenko, J. Voigt, O. Warnaar and W. Zudilin. The two final lectures on Friday were reports on the discoveries that had been done during the program duration: on critical L-values for products of up to 20 Bessel functions by D. Broadhurst and on a partial resolution of some motivic congruences by É. Delaygue (joint work with F. Beukers).

Left to right, rear to front: James Wan, Fang-Ting Tu, Yifan Yang, Éric Delaygue, John Voight, Adriana Salerno, Alex Ghitza, Mark Watkins, Masha Vlasenko, Wadim Zudilin, Piotr Achinger (with Helena), Jan de Gier, David Broadhurst, Ole Warnaar, Ravi Ramakrishna, Fernando Rodriguez Villegas, Sharon Frechette, Robert Osburn, Frits Beukers, Ling Long, David Roberts, Duco van Straten, Holly Swisher, Abdellah Sebbar. Courtesy of the MATRIX Institute, the University of Melbourne (2017).

Many projects were initiated during the three-week activity period in Creswick that are now in the process of development. It was instrumental to all the participants to have remarkable collaboration facilities during the program provided by the newly renovated MATRIX Institute building (the ex-building of forestry graduate students, for many years used just for storage). The work done by the MATRIX staff, under the leadership of its Director J. de Gier who happened to be a participant of our program, to organise the things as they are now, is tremendous and deserves praise.

The hospitality of the place underwent cultural and nature checks during the weekends and free Wednesday afternoons: the participants visited attractions in the neighbouring Ballarat, Daylesford and Hepburn, some came as far as to Melbourne, and most enjoyed the Great Ocean Road trip. The 26th of January was a

day of usual activity for the program, though the evening was truly in the spirit of the Australia Day and included a BBQ dinner. There were also wine-and-cheese welcomes on Tuesdays and an official program dinner on Thursday 12 January.

In spite of seasonal fire danger, the participants felt quite safe from this perspective. Our only familiarity with the topic was through sharing the campus facilities together with a huge group of CFA (Country Fire Authority) students for a couple of days. This uneasy experience was alleviated by the great support and assistance of the professional staff on Creswick campus. We particularly acknowledge the fantastic work of Adam Crutchfield, the Chef, and Lyn Wilks, the MATRIX Officer in residence, for numerous gastronomic and cultural memories.

The principal sponsor of the three-week mathematics event was the MATRIX Institute itself; it provided us with a substantial budget for accommodation, seminar venues and catering. We further acknowledge the wealthy sponsorship of the Foundation Compositio Mathematica, the National Science Foundation (#1642598) and the Number Theory Foundation. We are also grateful to the Australian Mathematical Sciences Institute, Australian Mathematical Society, CARMA of the University of Newcastle and the European Mathematical Society for the travel support of the workshop keynote speakers.

We greatly encourage our colleagues, from both Australia and overseas, to consider the MATRIX in Creswick as your next destination for collaborative research. We wish the MATRIX Institute a prosperous mathematical future.