Puzzle Corner

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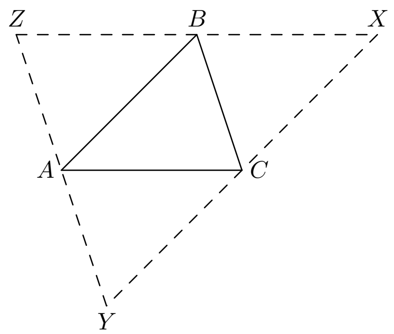
Solutions to Puzzle Corner 44

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 44 is awarded to Joe Kupka. Congratulations!

Triangular territory

Consider a finite set of points in the plane. Suppose that the area of the triangle formed by any three points is at most 1*. Prove that the entire set of points must lie in a triangle whose area is at most* 4*.*

Solution by Dave Johnson: Let the triangle with the largest area be ABC. Construct a line through A which is parallel to BC, a line through B which is parallel to AC, and a line through C which is parallel to AB. Let the resulting larger triangle be XYZ as shown in the diagram below.



Consider any other point D in the set. Since the area of ACD must be smaller than or equal to the area of ABC, the point D must lie below the line XZ. Similarly, the point D must lie to the right of the line YZ and to the left of the line XY. Therefore the point D must lie inside triangle XYZ. Since this argument holds for any point D in the set, it follows that the entire set of points must lie within triangle XYZ. Finally, since the area of XYZ is exactly four times the area of ABC which is no more than 1, the entire set of points must lie within a triangle whose area is at most 4, as required.

Perpendicular cuts

Let an irregular pizza *be a region in the plane which is closed, bounded and has a well-defined area. Prove that every irregular pizza can be cut into four pieces of equal area using two straight and mutually perpendicular cuts.*

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Solution by Joe Kupka: For convenience, suppose the pizza has a total area of 4. First note that for any given direction, it is possible to draw a *half-line* in that direction which divides the pizza into two pieces of equal area. This can be achieved by shifting the line sideways and applying the intermediate value theorem. Now draw a horizontal half-line and a vertical half-line. They will divide the pizza into four pieces. Let the top right piece have area $1 + a$, then the top left piece must have area $1 - a$. Let $f(0) = 1 + a$.

Now for $\theta \in (0, \pi/2)$, define $f(\theta)$ in a similar fashion. Draw a half-line which forms an anti-clockwise angle of θ with respect to the horizontal direction, and another half-line which forms an anti-clockwise angle of θ with respect to the vertical direction. The two half-lines again divide the pizza into four pieces. Let $f(\theta)$ be the area of the piece which extends upwards indefinitely. Finally, define $f(\pi/2) = 1-a$.

Roughly speaking, as θ varies from 0 to $\pi/2$, $f(\theta)$ is the area of the pizza within a rotating quadrant. We claim f is a continuous function on $[0, \pi/2]$. Indeed, since the pizza is bounded, it is contained within a circle of diameter d. It is easy to check that

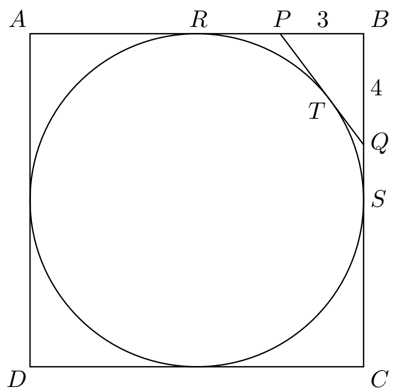
$$|f(\theta + h) - f(\theta)| \leq hd^2,$$

which implies continuity. By the intermediate value theorem, there exists an angle $\theta \in [0, \pi/2]$ such that $f(\theta) = 1$. Therefore the half-lines at that value of θ cut the pizza into four pieces of equal area, as required.

Inscribed radius

Let $ABCD$ be a square with an inscribed circle. Let P and Q be points on sides AB and BC, respectively, such that PQ is tangent to the circle. If $PB = 3$ and $QB = 4$, what is the radius of the circle?

Solution by Alan Jones: Let the midpoint of AB and BC be R and S respectively. Let the tangent point of PQ with the circle be T.



Since the two tangents from a point to a circle have equal length, we have the following equalities:

$$BR = BS, \qquad PR = PT, \qquad QT = QS.$$

Combining them with the fact that BR and BS equal to the radius r, we obtain:

$$\begin{aligned} 2r = BR + BS &= BP + PR + BQ + QS = BP + PT + TQ + QB \\ &= BP + PQ + QB = 3 + 5 + 4 = 12. \end{aligned}$$

Hence the required radius is given by $r = 6$.

Friendly division

Any two people are either friends or not friends. Given a group of people, is it always possible to divide them into two groups such that for any person, at least half of his/her friends are in the opposite group?

Solution by Joe Kupka: Yes it is possible. Divide them into two groups in a way which maximises the number of inter-group friendships. This is possible since the number of ways to divide everyone into two groups is finite. Now consider any person P. Suppose that less than half of P's friends are in the opposite group. But then we can simply move P to the opposite group to increase the total number of inter-group friendships. This is a contradiction. Thus at least half of Ps friends are in the opposite group in the first place. Since this holds for any person P, the required condition is therefore satisfied.

Squaring off

(i) *Amy and Bob are playing a game on an unmarked $n \times n$ chessboard. Amy begins by marking a corner square. Then Bob marks an unmarked square which is adjacent to (sharing an edge with) the square Amy just marked. Then Amy marks an unmarked square which is adjacent to the square Bob just marked. Then it is Bob's turn again and so on. This process continues until one of them can no longer make a valid move and loses the game. Who has a winning strategy.*

(ii) *If Amy's first move is to mark a square adjacent to a corner square, who has the winning strategy?*

Solution by Jensen Lai: (i) If n is even, the board can be divided into 1×2 blocks. Every time Amy marks one square of a 1×2 block, Bob can then mark the second square of the same block. This is a winning strategy for Bob as he will always have an available adjacent square to mark.

Now consider the case where n is odd. After Amy marks one of the corner squares, the rest of the board can once again be divided into 1×2 blocks. This time the roles are reversed. After Amy's first move, whenever Bob marks one square of a 1×2 block, Amy can then mark the other square of the same block. So in this case, Amy has the winning strategy.

(ii) If n is even, Bob can still use the winning strategy from part (i). If n is odd, Bob has a winning strategy as well in this case. Suppose Amy's first move is marking a square X adjacent to a corner square Y. By using a simple parity argument (chessboard colouring), Amy will never be able to mark Y. So Bob can simply pretend that he has already marked Y just before Amy's first move on X. Now Bob can use Amy's winning strategy from part (i) to guarantee a win.

Solutions to Puzzle Corner 45

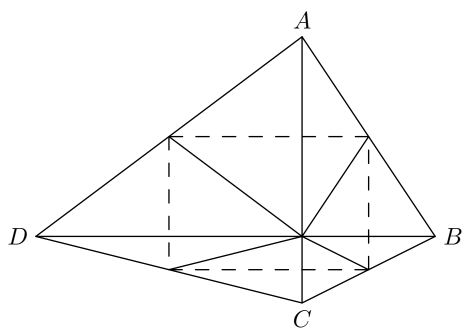
Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 45 is awarded to Jensen Lai. Congratulations!

Folding quadrilaterals

Find all quadrilaterals such that it is possible to fold all the corners neatly into a common point with no gaps or overlaps.

Solution by Alan Jones: This is possible if and only if the diagonals of the quadrilateral are perpendicular. Let the quadrilateral be $ABCD$. In order to fold the points A and B onto the same point, the two folds must both pass through the midpoint of AB. Similarly, the same argument applies for the midpoint of BC. Hence when folding the corner B, the fold line must be the line joining the midpoints of AB and BC. After the fold, the corner B will coincide with the foot of perpendicular from B to AC. Using the same argument, the corner D will be folded onto the foot of perpendicular from D to AC. Since B and D must be folded onto the same point, AC must be perpendicular to BD.

It suffices to check that this is possible for all quadrilateral whose diagonals are perpendicular. This is indeed possible by folding all four corners to the intersection of the diagonals.



Note that these folds are possible even if the quadrilateral is not convex. This can be done by first folding the three non-reflex corners onto the intersection of the diagonals, then folding that common point onto the reflex angle.

Summing strategy

There are 100 cards arranged in a row on the table. Each card is showing a positive integer. Two players now play a game. On each player's turn it is permitted to take either the rightmost or the leftmost card. This is done until all cards are taken. The winner is the player who has the greatest sum of numbers on his/her cards.

It is known that the sum of all cards equals 2015. Who has a winning strategy?

Solution by Dave Johnson: The first player has a winning strategy. Colour the cards so they alternate between red and blue. Now sum all the red cards and all the blue cards separately. The two sums must be different since the total sum is odd. Without loss of generality, suppose the red sum is greater. The first player can win by taking a red card every turn. This is possible since there is always an even number of cards just before the first player's move and thus the two end cards always have different colours. On the other hand, the second player must always take a blue card since both available end cards are always blue. When the game finishes the first player has all of the red cards and the second player has all of the blue cards. Therefore the first player wins since the red sum is greater than the blue sum.

Train tracks

Terrence is playing with toy train tracks and has constructed a closed circuit which contains no intersections and no linear segments. He used a large number of congruent standard rails, each having the shape of a quarter of a circle. Prove that the total number of tracks used is a multiple of 4.

Solution by Jensen Lai: Let the radius of curvature of the tracks be 1 unit. Suppose the circuit starts at the origin of a Cartesian plane and the first track begins by travelling vertically in the direction of the positive y axis. Each track toggles the direction of the circuit between vertical and horizontal. Therefore, if the circuit returns to the origin in the vertical direction, there must be an even number of tracks.

Now split the circuit into pairs of adjacent tracks. If a pair of tracks both turn in the same direction, clockwise or anticlockwise, then they form a U shape. Each U shape toggles the direction of the circuit between up and down. However, a U shape does not alter the y value of the circuit. If a pair of tracks turn in opposite directions, they form an S shape. Each S shape toggles the y value of the circuit between 0 (mod 4) and 2 (mod 4). However, an S shape does not alter the direction of the circuit. In order for the circuit to return to the origin in the starting direction, there must be an even number of S pairs and an even number of U pairs. Therefore, the total number of tracks is a multiple of 4.

Card array

Prove that if you deal out a standard deck of 52 cards into 4 rows of 13, then it is always possible to pick one card from each column to obtain 13 different card values. Note that the 13 cards do not have to have the same suit.

Solution: The solution uses Hall's marriage theorem, which is as follows. *Let there be n women and n men. For each woman, there is a subset of the men, any one of which she would happily marry; and any man would be happy to marry a woman who wants to marry him. For every subset S of women, let $f(S)$ be the subset of men whom at least one of the women would be happy to marry. If $|S| \leq |f(S)|$ holds for every subset S of women, then there it is possible to form n pairs of happily married couples.*

In the current problem, let there be 13 men denoted by the card values $A, 2, 3, \ldots, 10, J, Q, K$. Let there be 13 women corresponding to the 13 columns of the card array. For each woman, let the cards in the corresponding column represent the men whom she is happy to marry. We now check that the condition of Hall's marriage theorem is satisfied. Indeed, in any subset of k columns, there are $4k$ cards. There must be at least k different card values present in these $4k$ cards, as there are at most four occurrences of the same card value. Therefore Hall's marriage theorem can be applied here, pairing each column to a different card value contained within the column. This completes the solution.

Droid drivers

Larry and Rob are two robots travelling in a car from Arcadia to Zooland. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every l kilometres; Rob makes a 90° right turn after every r kilometres, where l and r are positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Given that the robots started from Arcadia facing the correct direction towards Zooland, for which choices of the pair (l, r), are they guaranteed to reach Zooland, regardless of how far it is?

Solution: If $\gcd(l, r) = g > 1$, then we can scale the variables in the problem by a factor of g. So it suffices to focus on the case where $\gcd(l, r) = 1$. We claim the car will always be able to reach Zooland if and only if $l \equiv r \pmod 4$. Note that the coprime condition implies that both l and r are either 1 or 3 $\pmod 4$.

For simplicity, we will position Arcadia at $(0, 0)$ and Zooland at $(d, 0)$, so the car starts out facing east. Let us consider the path of the car in *sections* of lr kilometres. It is clear that the car will have identical behaviour for each section.

First, let us eliminate the cases where $l \not\equiv r \pmod 4$.

- $l - r \equiv 2 \pmod 4$: After the first section, we have made l left turns and r right turns, which is equivalent to a net of two right turns. Let the displacement vector for the first section be (x, y). Since the car has rotated 180°, the displacement vector for the second section will be $(-x, -y)$, which will take it back to $(0, 0)$ and the car will be facing east again. We have returned to the starting configuration and the car has certainly never travelled further than $2lr$ kilometres from the origin. Hence it is not possible to reach Zooland if $d > 2lr$.

- $l - r \equiv 3 \pmod 4$: After the first section, we have made a net of one right turn. Let the displacement vector for the first section be (x, y) again. This time the car has rotated 90° clockwise. We can see that the displacement vectors for the second, third and fourth section will be $(y, -x)$, $(-x, -y)$ and $(-y, x)$ respectively. So after four sections, we are back at $(0, 0)$ and facing east again. Hence it is not possible to reach Zooland if $d > 4lr$.
- $l - r \equiv 1 \pmod 4$: This is similar to the previous case.

This leaves us with the case of $l \equiv r \pmod 4$. Here the car makes a net turn of 0° after each section of lr kilometres, and so it must be facing east. In order to reach Zooland, the card must traverse the entire positive x axis. We claim that the car will be at $(1, 0)$ after one section. Denote the kth kilometre of movement by m_{k-1}, which takes values from the complex numbers $1, i, -1$ or $-i$, depending on the direction. It suffices to prove $\sum_{k=0}^{lr-1} m_k = 1$.

For $l \equiv r \equiv 1 \pmod 4$. Define the following sequences for $k = 0, 1, 2, \ldots, lr - 1$:

$$p_k = (-i)^{k - \lfloor k/l \rfloor}, \qquad q_k = i^{k - \lfloor k/r \rfloor}.$$

In particular, p_k is a sequence with period l and the first l terms are $1, -i, -1, i, \ldots, 1$, while q_k is a sequence with period r and the first r terms are $1, i, -1, -i, \ldots, 1$. Furthermore, the product $p_k q_k$ also equals to the movement of the car m_k:

$$p_k q_k = (-i)^{k - \lfloor k/l \rfloor} \times i^{k - \lfloor k/r \rfloor} = i^{\lfloor k/l \rfloor} \times (-i)^{\lfloor k/r \rfloor} = m_k.$$

This holds as $\lfloor k/l \rfloor$ and $\lfloor k/r \rfloor$ are the exact number of left and right turns before the $(k + 1)$th kilometre.

Using the fact that p_k, q_k are periodic, we have $m_k = p_{(k \bmod l)} q_{(k \bmod r)}$. As l and r are coprime, by the Chinese Remainder Theorem, there is a bijection between pairs $(k \bmod l, k \bmod r)$ and the numbers $k = 0, 1, 2, \ldots, lr - 1$. Hence we have:

$$\sum_{k=0}^{lr-1} m_k = \sum_{k=0}^{l-1} p_k \sum_{k=0}^{r-1} q_k = (1 - i - 1 + i + \cdots + 1)(1 + i - 1 - i + \cdots + 1) = 1$$

as required.

For $l \equiv r \equiv 3 \pmod 4$, the same argument can be applied but with slightly different definitions of p_k and q_k:

$$p_k = i^{k + \lfloor k/l \rfloor}, \qquad q_k = (-i)^{k + \lfloor k/l \rfloor}.$$

Once again we have $\sum_{k=0}^{lr-1} m_k = \sum_{k=0}^{l-1} p_k \sum_{k=0}^{r-1} q_k = 1$.

So we have proven that, after a section of lr kilometres, the car is indeed at $(1,0)$ facing east. In fact, the car has traversed the entire line segment joining $(0,0)$ and $(1,0)$ in its first move of the section. Therefore, given $l \equiv r \pmod 4$, the car will reach Zooland by the time (most likely before) it has travelled $\lfloor d \rfloor lr + 1$ kilometres.

Ivan is a Research Fellow at Monash University. His research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.