Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 45. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to `ivanguo1986@gmail.com` or send paper entries to: Gazette of the Australian Mathematical Society, Faculty of Science and Technology, Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 45 is 15 January 2016. The solutions to Puzzle Corner 45 will appear in the March 2016 issue of the *Gazette*. This will also be my last Puzzle Corner, after which the column will be on hiatus. I would like to thank those who have read, solved and contributed puzzles over the last six years, as well as the *Gazette* editors for their continuing efforts.

Folding quadrilaterals

Find all quadrilaterals such that it is possible to fold all the corners neatly into a common point with no gaps or overlaps.

Summing strategy

There are 100 cards arranged in a row on the table. Each card is showing a positive integer. Two players now play a game. On each player's turn it is permitted to take either the rightmost or the leftmost card. This is done until all cards are taken. The winner is the player who has the greatest sum of numbers on his/her cards.

It is known that the sum of all cards equals 2015. Who has a winning strategy?

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Train tracks

Photo: Martin Hotchin

Terrence is playing with toy train tracks and has constructed a closed circuit which contains no intersections and no linear segments. He used a large number of congruent standard rails, each having the shape of a quarter of a circle. Prove that the total number of tracks used is a multiple of 4.

Card array

Prove that if you deal out a standard deck of 52 cards into 4 rows of 13, then it is always possible to pick one card from each column to obtain 13 different card values. Note that the 13 cards do not have to have the same suit.

Droid drivers

Larry and Rob are two robots travelling in a car from Arcadia to Zooland. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every l kilometres; Rob makes a 90° right turn after every r kilometres, where l and r are positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Given that the robots started from Arcadia facing the correct direction towards Zooland, for which choices of the pair (l, r), are they guaranteed to reach Zooland, regardless of how far it is?

Solutions to Puzzle Corner 43

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 43 is awarded to Adrian Nelson. Congratulations!

Floating fedora

Sammy dives from a bridge into a river and swims upstream for one hour at constant speed. She then turns around and swims downstream at the same speed. As Sammy passes under the original bridge, a bystander tells her that her hat fell into the river the moment she dived into the water. In order to retrieve her hat, Sammy continues to swim downstream at the same speed. She finally catches up to her hat when she is exactly one kilometre away from the bridge. Assuming it is constant, what is the speed of current?

Solution by Steve Clarke: The answer is 0.5 km per hour. Consider the problem from the perspective of the hat. Note that the relative position of Sammy with respect to the hat is not affected by the speed of the current since they are both in the water.

Sammy swam away from the hat for one hour, so it takes another hour for her to swim back to the hat. In total, the hat has moved 1 km from the bridge in two hours. Therefore the speed of the current is 0.5 km per hour.

Social network

There is a group of 300 Twitter users, such that each one is following exactly one other person in the group. Prove that it is possible to find a smaller group of 100 in which no one is following anyone else.

Solution by Dave Johnson: We will solve the following generalisation:
'Suppose there is a set S of n twitter users, such that each one is following exactly one other person. Prove that it is possible to find a subset of size $n/3$ in which no one is following anyone else.'

We proceed via induction. The claim be easily checked for $n \leq 5$. For $n \geq 6$, there are two cases.

(i) There exists a person y who is followed by no one. Suppose that y follows x. Consider the set $S \setminus \{x, y\}$. By the inductive hypothesis, there exists a subset T of size $\lceil (n-2)/3 \rceil$ in which no one is following anyone else. Then the set $T \cup \{y\}$ satisfy the required conditions since

$$\left\lceil \frac{n-2}{3} \right\rceil + 1 = \left\lceil \frac{n+1}{3} \right\rceil > \frac{n}{3}.$$

(ii) Everyone is followed by someone else. In this case the set S can be decomposed into disjoint cycles of followers. Now it suffices to show that we can choose at least $1/3$ of the people from each cycle satisfying the required condition. This can be easily achieved by selecting every second person in the cycle, with the exception of leaving out the final person if the cycle length is odd.

This completes the induction and the solution.

Digit divisibility

A number is said to be elegant *if its digit sum is divisible by eleven. How many elegant numbers are there in the set* $\{0, 1, 2, \ldots, 10^{11} - 1\}$*?*

Solution by Adrian Nelson: We show there are $\frac{1}{11}(10^{11} - 10)$ elegant numbers in $\{0, 1, 2, \ldots, 10^{11} - 1)\}$.

By the uniqueness of the decimal expansion, every number in the set can be written as an 11-tuple $(a_0, \ldots, a_{10})$ of digits chosen from $\{0, 1, 2, \ldots, 9\}$. The number of such sequences with digit sum n is the coefficient of x^n in the expansion of

$$F(x) = (1 + x + \cdots + x^9)^{11}.$$

So it suffices to find the sum of the coefficients of x^n where n is divisible by 11.

Let ζ be a primitive 11th root of unity. In particular, $\zeta^{11} = 1$ and $\zeta \neq 1$. Furthermore,

$$1 + \zeta^n + \zeta^{2n} + \cdots + \zeta^{10n} = \begin{cases} 0, & \text{if } 11 \nmid n, \\ 11, & \text{if } 11 \mid n. \end{cases} \tag{1}$$

Now consider the expansion of

$$G(x) = \frac{1}{11}(F(x) + F(\zeta x) + F(\zeta^2 x) + \cdots + F(\zeta^{10} x)).$$

Denote the coefficient of x^n in $F(x)$ by f_n. Applying (1), the coefficient of x^n in $G(x)$ is given by

$$\frac{1}{11}(f_n + \zeta^n f_n + \zeta^{2n} f_n + \cdots + \zeta^{10n} f_n) = \begin{cases} 0, & \text{if } 11 \nmid n, \\ f_n, & \text{if } 11 \mid n. \end{cases}$$

Thus the required sum is simply the sum of the coefficients in $G(x)$, or

$$G(1) = \frac{1}{11} \sum_{m=0}^{10} F(\zeta^m).$$

The $m = 0$ summand is given by $F(1) = 10^{11}$. For each of $m = 1, 2, \ldots, 10$, (1) implies

$$F(\zeta^m) = (1 + \zeta^m + \cdots + \zeta^{9m})^{11} = (-\zeta^{10m})^{11} = -1.$$

Therefore the total number of elegant numbers is $G(1) = \frac{1}{11}(10^{11} - 10)$.

Square solitaire

Four pegs are initially placed on the ground so that they form a square. At each move, you may take an existing peg from some point P and move it to a new point P', as long as there is another peg at the midpoint of PP'. Is it possible to form a larger square using the four pegs after a finite number of moves?

Solution by John Butcher: The answer is no, it is not possible to form a larger square. First note that the inverse of each possible move is also a valid move. If a finite number of moves can lead to a larger square, then the inverses of these moves carried out in reverse order will lead to a smaller square.

Suppose the four pegs start on the coordinates $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$, it is not possible to form a smaller square since the pegs will always be on integer coordinates. This gives the required contradiction.

Rational coordinates

Does there exist a sphere (i.e. the surface of a ball) in $\mathbb{R}^3$, such that exactly one point on it has only rational coordinates?

Solution by Jensen Lai: The answer is yes. Consider a sphere of radius $\sqrt{2}$ centred at $(0, 0, \sqrt{2})$. The equation of the sphere is given by

$$x^2 + y^2 + (z - \sqrt{2})^2 = 2,$$

which rearranges to

$$x^2 + y^2 + z^2 = 2\sqrt{2}z. \tag{2}$$

Let (x, y, z) be a point on the sphere with rational coordinates, satisfying (2). The left-hand side of (2) is rational, which implies that $2\sqrt{2}z$ is rational. This is only possible if $z = 0$. Substituting back into (2) yields $x^2 + y^2 = 0$, so $x = y = 0$. Therefore $(0, 0, 0)$ is the only point with rational coordinates on the sphere, as required.

Ivan is a Postdoctoral Research Associate in the School of Mathematics and Applied Statistics at The University of Wollongong. His research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.