Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 41. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, Faculty of Science and Technology, Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 41 is 1 May 2015. The solutions to Puzzle Corner 41 will appear in Puzzle Corner 43 in the July 2015 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Improbable product

Is it possible for the product of four consecutive positive integers to be equal to the product of two consecutive positive integers?

Many folds

Submitted by Andrew Kepert

(i) An A4 paper has the length to width ratio of $\sqrt{2} : 1$. How many folds are needed to locate a point on the longer edge that divides the edge into the ratio of $1 : 3$?

(ii) Start with a rectangular piece of paper, choose an edge and mark a point somewhere along it. Now there are two 'far' corners which do not belong

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to the chosen edge. Make a fold so that one of these far corners coincides with the marked point, then unfold. Make another fold so that the other far corner coincides with the marked point, then unfold again. Prove that the intersection point of the two creases has equal distance to two opposite edges of the paper.

Rabbit season

(i) Rachel and Fran are playing a game. Rachel controls three 'rabbit' pieces, while Fran controls a single 'fox' piece. Initially, all four pieces are placed somewhere along a straight line. They take turns making moves, with Rachel going first. Each move, a player is allowed to move one of her pieces a distance of at most one unit along the straight line. Fran wins if her fox piece can catch one of the rabbit pieces. Can Fran always win?

(ii) The same game is now played on a two-dimensional plane instead of a straight line. The rules are the same, except now Rachel has 20 'rabbit' pieces. Can Fran always win?

Lighthouse logic

There are 18 fixed lighthouses in the plane, each has the ability to illuminate an angle of 20°. Prove that, by carefully selecting the directions in which the lighthouses are operating, it is always possible to illuminate the whole plane.

Solutions to Puzzle Corner 39

Many thanks to everyone who submitted. The \$50 book voucher for the best submission to Puzzle Corner 39 is awarded to Dave Johnson. Congratulations!

Integral means

Given n positive integers $a_1, a_2, \ldots, a_n$, their arithmetic, geometric and harmonic means are defined as follows:

$$\begin{aligned} \textit{arithmetic mean} &= \frac{a_1 + a_2 + \cdots + a_n}{n}, \\ \textit{geometric mean} &= \sqrt[n]{a_1 a_2 \cdots a_n}, \\ \textit{harmonic mean} &= \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}. \end{aligned}$$

Can you find n distinct *positive integers such that their arithmetic, geometric and harmonic means are also positive integers?*

Solution by Joe Kupka: Yes it is possible for all three means to be positive integers. The key observation is that if we scale all of the numbers by a constant factor, all three means are also scaled by the same factor. Since the arithmetic and harmonic means are automatically rational numbers, it suffices to begin with a set of number whose geometric mean is an integer and then scale accordingly.

An example would be $a_i = (n^{2n} - 1)n^{2i}$ for $1 \le i \le n$. In this case the three means are given by:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = (n^{2n} - 1)\frac{n^2 + n^4 + \cdots + n^{2n}}{n} = \frac{n(n^{2n} - 1)^2}{n^2 - 1},$$

$$\sqrt[n]{a_1 a_2 \cdots a_n} = (n^{2n} - 1)\sqrt[n]{n^{2+4+\cdots+2n}} = (n^{2n} - 1)n^{n+1},$$

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \frac{n(n^{2n} - 1)}{n^{-2} + n^{-4} + \cdots + n^{-2n}} = (n^2 - 1)n^{2n+1}.$$

They are indeed all positive integers.

Products of sums

We are given an $n \times n$ table where n is odd. An odd integer is written in each of its squares. Is it possible for the product of the column sums to be the negative of the product of the row sums?

Solution by Jensen Lai: The answer is no. Consider the problem in modulo 4. Each of the entries can be replaced by either 1 or -1. Since n is odd, the column sums, row sums and the corresponding products are all congruent to either 1 or -1 in modulo 4. We shall show that the product of the column sums is always congruent to the product of the row sums. This is certainly true if every entry is 1.

Let us study the effect of changing one entry from 1 to -1. One row sum will switch between 1 and -1. So the product of the row sums will also switch between 1 and -1. The same is true for one column sum as well as the product of the column sums. Now every possible table can be reached by changing some entries from 1 to -1, one at a time. Afterwards, both the product of the column sums and the product of the row sums will have switched the same number of times between 1 and -1. This implies that the two products are indeed always congruent in modulo 4.

However, if we add the product of the column sums with the product of the row sums, the result will always be congruent to 2 in modulo 4, which is certainly not 0. Therefore the two products cannot be negatives of each other.

Negative base

Given a positive integer $b > 1$, a base $-b$ representation *of a number n refers to the following form:*

$$n = a_0(-b)^0 + a_1(-b)^1 + \cdots + a_k(-b)^k$$

where $a_0, a_1, \ldots, a_k$ are non-negative integers less than b.

Prove that, for any positive integer $b > 1$, every integer (not just positive) has a unique base $-b$ representation.

Solution by Dave Johnson: We begin with an auxiliary result:

Fix a positive integer $b > 1$. If $-b < x_0, x_1, \ldots, x_k < b$, then the equation

$$0 = x_0(-b)^0 + x_1(-b)^1 + \cdots + x_k(-b)^k$$

is satisfied if and only if $x_0 = x_1 = \cdots = x_k = 0$.

By considering the equation in modulo b, we see that $x_0 = 0$. Now divide both sides by $-b$ and repeat the argument to obtain $x_1 = 0$ and so on. This proves the result. An immediate corollary is that any integer n can have *at most one* base $-b$ representation, because otherwise the difference between two representations will contradict the result above.

Back to the problem at hand. Consider all of the possible base $-b$ representations up to order k. In particular, we refer to

$$n = a_0(-b)^0 + a_1(-b)^1 + \cdots + a_k(-b)^k,$$

where $0 \leq a_0, a_1, \ldots, a_k < b$. It is clear that the terms with even powers are non-negative while the terms with odd powers are non-positive. This can be used to create upper and lower bounds for n, given by

$$(b-1)(-b)^1 + (b-1)(-b)^3 + \cdots \leq n \leq (b-1)(-b)^0 + (b-1)(-b)^2 + \cdots,$$

with the exponents bounded by k. The number of integers contained in this closed interval is given by

$$1 + (b-1)(b^0 + b^1 + \cdots + b^k) = b^{k+1}.$$

On the other hand, since each of $a_0, a_1, \ldots, a_k$ can take b possible values, there are exactly b^{k+1} possible base $-b$ representations up to order k. Recall that every integer has at most one base $-b$ representation, it follows that every number in this range has *exactly one* such base $-b$ representation. As k gets larger, it is clear that the upper and lower bounds grow indefinitely in both directions. Therefore every integer has a unique base $-b$ representation.

Ranking matches

(i) *Four table tennis enthusiasts are gathered to pit their skills against one another. There is a clear order in their table-tennis abilities and the better player always wins in a match. How many matches are needed to rank everyone according to their skill levels?*

(ii) *What if there were five table tennis enthusiasts to begin with?*

Solution: The answers to parts (i) and (ii) are 5 and 7, respectively. Out of n people, there are $n!$ possible ordering of skill levels. Since each match can have two possible outcomes, a minimum of $\lceil \log_2(n!) \rceil$ matches is required to determine a complete ordering.

For 4 people in part (i), at least $\lceil \log_2(4!) \rceil = 5$ matches are needed. This is achievable via a standard 'double elimination' format. Denote the players by A, B, C and D.

1. First, A plays against B, while C plays against D. Without loss of generality, suppose A and C are the winners.

2. Then, A plays against C in the winners' match, while B plays against D in the losers' match. Without loss of generality, suppose A wins and takes first place, while D loses and takes last place.
3. Finally, let B play against C to determine second and third places.

For 5 people in part (ii), at least $\lceil \log_2(5!) \rceil = 7$ matches are needed. This turns out to be possible as well, although the procedure is more complicated. Denote the players by A, B, C, D and E.

1. First, A plays against B, while C plays against D. Without loss of generality, suppose A and C are the winners.
2. Then, let A play C in the winners' match. Without loss of generality, suppose A wins. Up to this point, we have determined that $A > B$ and $A > C > D$.
3. Now, we determine the position of E within the $A > C > D$ chain. This can be achieved in two matches. First let E play against C. If E wins, then let him play against A. Otherwise let him play against D. After this, we have a complete ordering of A, C, D and E.
4. Finally we have to find the position of B using only two more matches. So far we only have $A > B$. There are two cases. If the previous step produced $E > A > C > D$. Then we can simply play B against C and D to complete the ordering. If $A > E$ occurs instead, then we may, without loss of generality, assume that $A > E > C > D$ since none of E, C or D have played against B. Now we can simply repeat the method used in the previous step to find the position of B amongst $E > C > D$, by first matching B against C, then E or D depending on the outcome.

In all cases, we have determined the complete ordering in 7 matches.

Note: As it turns out, the bound of $\lceil \log_2(n!) \rceil$ is not always achievable. The smallest counter-example occurs when $n = 12$, where $\lceil \log_2(12!) \rceil = 29$ but 30 matches are required. In general, determining the exact number of matches needed is computationally difficult and no simple formula is known.

Circular cuts

Submitted by Ross Atkins. The magician announces his next trick. "I have here, a piece of cardboard in the shape of a perfect circle. For my next act, I shall cut it into a number of pieces, so that all the pieces are absolutely identical to each other in shape and size..."

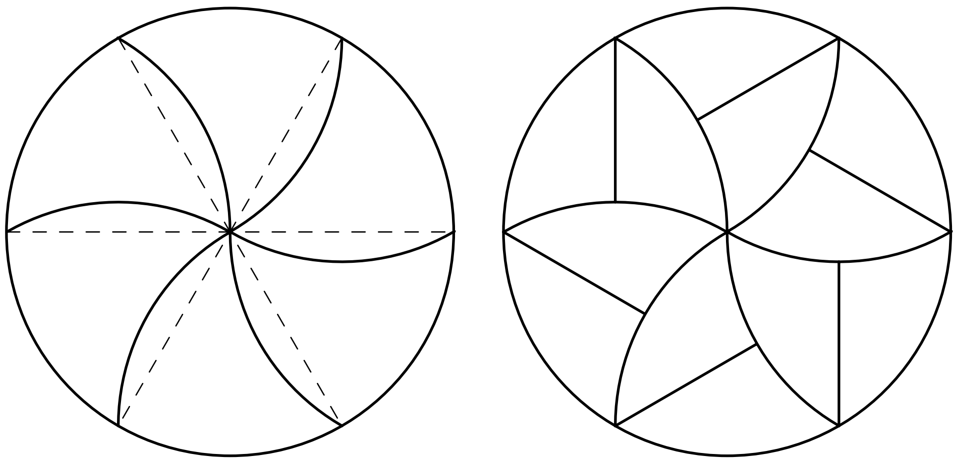
"So what", a restless audience member interjects, "anyone can do that, you've never seen a sliced pizza before?"

The magician keeps his composure. "Please let me finish. When I'm done cutting, at least one of the final pieces would not have touched the centre of the original circle to begin with."

Some started to scratch their heads. "Surely that's impossible!"

Will the magician be able to back up his words?

Solution by Jensen Lai: Yes, the amazing magician will be able to deliver. This is certainly one of those problems where pictures are worth more than words.



Ivan is a Postdoctoral Research Fellow in the School of Mathematics and Applied Statistics at The University of Wollongong. His current research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.