Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 40. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, Faculty of Science and Technology, Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 40 is 15 January 2015. The solutions to Puzzle Corner 40 will appear in Puzzle Corner 42 in the May 2015 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Rolling riddle

On average, how many times do you have to roll a die before all six numbers appear at least once?

Balanced views

Given a convex polygon $A_1A_2\cdots A_n$ in the plane, we say a point P (in the same plane) is *balanced* if

$$\angle A_1PA_2 = \angle A_2PA_3 = \cdots = \angle A_{n-1}PA_n = \angle A_nPA_1.$$

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(i) Prove that for any convex polygon with an odd number of sides, there is at most one balanced point in the plane.

(ii) Can there ever be more than one balanced point if the convex polygon has an even number of sides?

Spherical stroll

An ant is crawling on the surface of a sphere whose radius is one metre. After a while, the ant returns to its starting position. Prove that if the ant has crawled no more than 2π metres, then its path can be contained in some hemisphere of the sphere.

Digital division

Consider the set of all five-digit numbers whose decimal representation is a permutation of digits 1, 2, 3, 4 and 5. Is it possible to divide this set into two groups, so that the sum of the squares of the numbers in each group is the same?

Tricky triangulation

For $n \geq 3$, a convex n-gon can be divided into $n-2$ triangles by using $n-3$ of its diagonals. This is called a *triangulation*. For which values of n is it possible to triangulate a convex n-gon such that every vertex is adjacent to an odd number of the resulting triangles?

Solutions to Puzzle Corner 38

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 38 is awarded to Jensen Lai. Congratulations!

Surface temperature

For the purpose of this puzzle, let us assume that the Earth is perfectly spherical, and the surface temperature is a continuous function of the Earth's surface.

(i) *Prove that there exist two antipodal points with the same surface temperature.*

(ii) *Fix a distance d less than the diameter of the Earth. Prove that there exist two points exactly d apart, that have the same surface temperature.*

Solution by Aaron Hassan: (i) For any point X on the Earth's surface, define a function $f(X)$ to be the difference in surface temperature between X and its antipodal point X', so $f(X) = T(X) - T(X')$ where T is the surface temperature. Since $f(X)$ is a continuous function of the Earth's surface, it suffices to find a point X such that $f(X) = 0$.

Take any point P on the equator and suppose that $f(P) \neq 0$. Without loss of generality, say $f(P) > 0$. If P' is the antipodal point of P, then

$$f(P') = T(P') - T(P) = -f(P) < 0.$$

Since $f(P)$ and $f(P')$ have opposite signs, by the intermediate value theorem, as we move from P to P' along the equator, there must be a point Q at which $f(Q) = 0$. Thus there must exist two antipodal points with the same surface temperature.

(ii) The argument is similar to part (i). Instead of the equator, let us concentrate on a circle Γ of diameter d on the Earth's surface (e.g. a set of points with the same latitude). This is possible since d is less than the diameter of the Earth. For any point X on Γ, define the function $f(X)$ to be the temperature difference between X and its opposite point on Γ. The argument from part (i) can now be applied to find two diametrically opposite points on Γ (so they have distance d) with the same surface temperature.

Triangle existence

(i) *For which integer values of x does there exist a non-degenerate triangle with side lengths of* 5, 10 *and x?*

(ii) *In a triangle, an altitude length refers to the perpendicular distance from a vertex to the opposite side. For which integer values of x does there exist a non-degenerate triangle with altitude lengths of* 5, 10 *and x?*

Solution by M.V. Channakeshava: (i) First recall the triangle inequality, which states that in any non-degenerate triangle, the sum of the two shorter sides is strictly greater than the longest side. Conversely, it is always possible to construct a triangle with given side lengths as long as they satisfy the triangle inequality.

In the present context, we have the inequalities

$$5 + 10 > x, \qquad 5 + x > 10, \qquad 10 + x > 5.$$

These are equivalent to $5 < x < 15$. Therefore the possible integer values for x are 6, 7, 8, 9, 10, 11, 12, 13 and 14.

(ii) If the area of the triangle is A, then the three sides of the triangle must be $\frac{2A}{5}$, $\frac{2A}{10}$ and $\frac{2A}{x}$. Again applying the triangle inequality, we have

$$\frac{2A}{5} + \frac{2A}{10} > \frac{2A}{x}, \qquad \frac{2A}{5} + \frac{2A}{x} > \frac{2A}{10}, \qquad \frac{2A}{10} + \frac{2A}{x} > \frac{2A}{5}.$$

These simplify to $\frac{10}{3} < x < 10$. So the possible integer values for x are 4, 5, 6, 7, 8 and 9.

Since we are working with the altitudes instead of the side lengths, the existence of such a triangle is less straightforward. To construct a triangle with altitudes 5, 10 and x where $\frac{10}{3} < x < 10$, start by constructing a triangle with side lengths $\frac{1}{5}$, $\frac{1}{10}$ and $\frac{1}{x}$. The altitudes of this triangle have the correct ratio of $5 : 10 : x$, so we can simply scale the triangle until they have the correct lengths.

Colourful lattice

In the coordinate plane, points with integer coordinates are called lattice points*.*

(i) *Suppose that each lattice point is coloured using one of n possible colours. Prove that there exist four lattice points with the same colour which are also the vertices of a rectangle.*

(ii) *Suppose that each lattice point is either coloured using one of n possible colours, or not coloured at all. Furthermore, suppose that it is not possible to find four lattice points with the same colour which are also the vertices of a rectangle. Prove that there exist arbitrarily large squares such that none of lattice points in their interior is coloured at all.*

Solution by Joe Kupka: (i) Consider the following horizontal row of $n+1$ lattice points

$$R_m = \{(1,m),\ (2,m),\ \ldots,\ (n+1,m)\}.$$

Since there are $n+1$ points but only n colours, two points in R_m must have the same colour.

Consider the rows $R_1, R_2, R_3, \ldots$ over all possible values of m. Since there are infinitely many such rows but only n^{n+1} possible colour combinations, there must be two rows, say R_i and R_j, with identical colourings. Recall that two points in R_i, say (p,i) and (q,i), have the same colour. It follows that the points (p,i), (q,i), (p,j) and (q,j) form a monochromatic rectangle.

(ii) Let N be an arbitrarily large positive integer. We can divide the lattice points into $N \times N$ blocks, each containing N^2 lattice points. Now treat each block as a single *hyper-lattice-point*, coloured by one of $(n+1)^{N^2}$ *hyper-colours* based on the colouring of the N^2 original points in the block.

We can apply part (i) to our hyper-lattice-points and hyper-colours. This produces four blocks with identical colourings in the position of a rectangle. If any of the original lattice points inside these blocks are coloured at all, then the four corresponding points from the four blocks immediately form a monochromatic rectangle, which is a contradiction. Thus none of these $N \times N$ blocks can be coloured at all. Since the choice of N was arbitrary, there must exist arbitrarily large squares with uncoloured interior lattice points, as required.

Drawing parallels

Two parallel lines are drawn on a sheet of paper. There is also a marked point which does not lie on either of these lines. Here is your challenge: using only an unmarked straight edge (and no compass), construct a new line through the marked point, that is also parallel to the two existing lines.

Bonus: *Can you find two different ways to achieve this?*

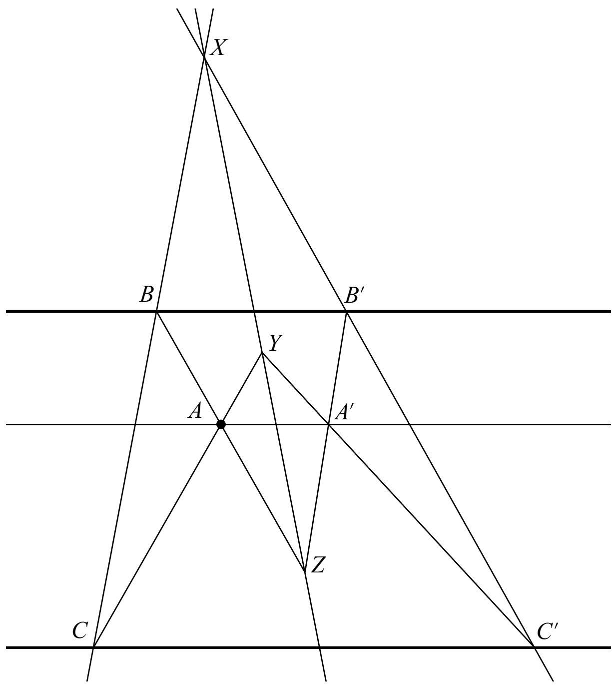
Solution by Jensen Lai: In this construction we shall utilise Desargues' theorem, which is as follows.

Let ABC and $A'B'C'$ be two triangles. Define the following points of intersection:

$$X := BC \cap B'C', \qquad Y := CA \cap C'A', \quad Z := AB \cap A'B'.$$

Then the points X, Y and Z are collinear if and only if the lines AA', BB' and CC' are either concurrent or parallel.

Refer to the diagram below. Suppose we would like to construct a line through A parallel to the two highlighted lines. Start by drawing three concurrent lines through an arbitrary point X and let two of them intersect the parallel lines at B, C and B', C' as shown. Now let BA and CA intersect the third line at the points Z and Y respectively. Finally let the intersection of YC' and ZB' be A'.



Since X, Y and Z are constructed to be collinear, we may apply Desargues' theorem to triangles ABC and $A'B'C'$. In particular, AA' must be parallel to BB' and CC', as required.

Note: There is an alternative solution to the problem that uses Pascal's theorem. The theorem is as follows.

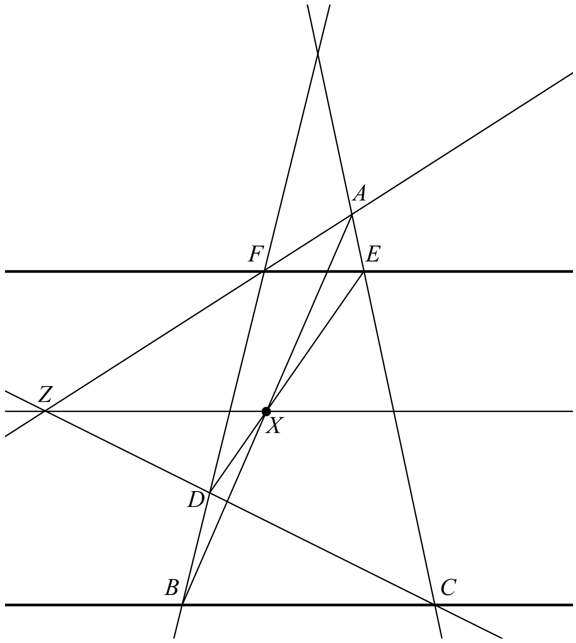
Let A, B, C, D, E and F be six points lying on the same conic. Define the following points of intersection:

$$X := AB \cap DE, \quad Y := BC \cap EF, \quad Z := CD \cap FA.$$

Then the points X, Y and Z are collinear.

Refer to the diagram below. We aim to construct a third parallel line through a point X this time. Begin by drawing two arbitrary lines and let them intersect the two parallel lines at F, B and E, C as shown. Now let D be the intersection of EX

and FB, and A be the intersection of BX and EC. Finally let the intersection of DC and AF be Z.



Since A, B, C, D, E and F lie on a pair of lines, which is a conic, we may apply Pascal's theorem. The three collinear points are X, Z and the intersection of BC and EF. But since BC and EF are parallel, we have to take the limiting case where their intersection Y is at infinity. The collinearity of X, Y and Z implies that XZ also passes through the same point at infinity. Therefore XZ is parallel to BC and EF, as required.

Ivan is a Postdoctoral Research Fellow in the School of Mathematics and Applied Statistics at The University of Wollongong. His current research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.