Book Reviews

Lectures on Real Analysis

Finnur Lárusson
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This slim volume is a real gem (pun intended). Starting from an axiomatic treatment of the real numbers — on which more later — it builds up a rigorous first treatment of real analysis. It works up to the point of being able to define and show the properties of exp, log, sin and cos, and proves, with the aid of a little metric space theory, Picard's theorem. This gives the unique existence of continuously differentiable solutions to certain first-order, single variable differential equations. The exposition is aided by many examples and exercises, which move between very concrete problems and calculations, and abstractly useful abstract lemmas. This book is currently the textbook for the second-year real analysis course at the University of Adelaide (alas that I did not get to enjoy this book as a student!).

Lárusson states the two goals of the book as: to treat calculus carefully from first principles and to practice the reading and writing of proofs. The exercises that deal with proofs start out from the level of proving standard properties of rational numbers from ordered field axioms, and basic naïve set theory. They progress with the material in sophistication up to, for instance, one direction of the Heine–Borel theorem, or the equivalence of the topological and metric definitions of continuity for maps between metric spaces. As a basis for a lecture course, the existence of so many exercises through the text will be welcomed by lecturers, especially as they are well integrated into the progression of each chapter.

The axiomatic base that the book starts from is that of a complete ordered field. Or rather, the Peano axioms for the natural numbers are implicitly assumed (in the guise of the induction axiom and the 'standard structure' of the naturals), as well as the existence of integers and rationals, and then the book introduces the axioms for a field and ordered field. Some basic set theoretic notions and terminology is also introduced, such as functions and boolean operations on sets. Two proofs are given for the non-existence of a rational square root of two, the first of which does not rely on the notion of divisibility and common factors, as should be the case in a ground-up treatment with no number-theoretic background.

Completeness, as an axiom, is introduced in Chapter 2, using the least upper bound property, and the real numbers are defined as 'the' complete ordered field. A real number is defined as 'an element of a complete ordered field'. Since any such field is unique up to an isomorphism of complete ordered fields, it does not matter which model one uses, or if one works *synthetically*: from the axioms with no reference to models. The existence of constructions (Cauchy sequences and Dedekind cuts) of the reals from the rationals are mentioned, but not treated. In the reviewer's opinion, this is not a drawback in a first course on analysis, since the details of these constructions are not actually used in anything that follows. The

Archimedean property and other standard properties, including uncountability, are shown to follow from completeness. The book uses Cantor's original proof, using nested intervals, rather than any explicitly diagonal argument. Using decimal expansions is a model-dependent approach and not without irrelevant subtleties!

The third chapter introduces sequences and series, and culminates in showing the equivalence of five characterisations of the field of real numbers: completeness, nested interval property plus the Archimedean property, the monotone convergence theorem, the Bolzano–Weiertrass theorem, and the Cauchy criterion plus the Archimedean property. This is a taste of the area called Reverse Mathematics: showing exactly which axioms are necessary, and usually equivalent to, a given theorem. Such a result is useful for emphasising just why the axioms we use are chosen.

Chapters 4 and 5 treat the basic topological notions of open, closed and compact sets, and continuity. The approach to continuity and compactness is via sequences, so one could argue that the book uses sequential contintinuity and sequential compactness throughout. For metric spaces, such as the reals or certain function spaces as used in the book later, sequential definitions are equivalent to the usual notions. We also meet, at this point, uniform continuity and the intermediate value theorem. One of the exercises in this chapter is to prove the existence of nth roots in the reals, fulfilling a promise made in Chapter 2. In fact, this is a pleasant feature of the book: sometimes concepts that are 'known' (existence and basic properties of the function sin, for instance) are used, but always with a forward pointer to where they will be proved in a later chapter.

Chapters 6 and 7 treat differentiation and (Riemann) integration respectively, covering the standard theorems (Rolle's theorem, mean value theorem, l'Hôpital's rule, fundamental theorem of calculus). Importantly, continuous functions on intervals are proved Riemann integrable, though it is pointed out that discontinuous functions may or may not be so (the last exercise gets the student to show that a certain function with uncountably many discontinuities is Riemann integrable). Chapter 7 ends with defining the logarithm and exponential functions.

Chapter 8 is in a sense the end of the 'core' part of the book, treating sequences and series of functions, power series, Taylor series and the like. The spaces of continuous and integrable functions are proved to be complete for the sup norm, though in more elementary terms. This is used in the later chapters dealing with function spaces as metric spaces. The trigonometric functions are approached as follows. Suppose we are given bounded smooth functions s and c such that $s' = c$, $c' = -s$, $s(0) = 0$ and $c(0) = 1$. Then from a corollary to Lagrange's remainder theorem (both given in the text), one can write down the Maclaurin series for both s and c, and show that they are uniquely determined by these properties. Clearly s and c will turn out to be sin and cos respectively, and some of their standard properties are derived. The existence of the number π is defined as the unique smallest period of sin (equivalently one could say the smallest positive zero of sin).

The last two chapters are a relaxed introduction to metric spaces and continuous maps, enough to treat the metric spaces $\mathcal{C}[a,b]$ with the sup metric. Plenty of other important examples are given to show the ubiquity of the notion of metric (sequence spaces, ultrametric spaces, Euclidean spaces with various standard metrics). The key result of the last chapter, leading up to Picard's theorem, is the Banach fixed-point theorem, stating that to every continuous contraction of a non-empty complete metric space, there is a unique point fixed by that contraction.

The book is described in the back-matter as being appropriate for a second-year undergraduate, or a more advanced student needing a foundation in real analysis. This should not be its only audience; anyone writing a textbook should have a copy as an example in how to do the job well. The scope of the book is ideal, and it is a pity that there are not more undergraduate texts of this size and quality.

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Lectures on $N_X(p)$

Jean-Pierre Serre
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Jean-Pierre Serre requires little introduction. His mathematical contributions were recognised in 1954 by a Fields Medal and in 2003 by the inaugural Abel Prize (and by many other awards in-between), while his exceptional writing skills were rewarded in 1995 by a Steele Prize for mathematical exposition. The book under review, based on lectures delivered in 2009 and 2011 in Taiwan and thoroughly revised for publication, lives up to the high expectations created by the juxtaposition of technical prowess and presentation genius that characterises Serre's work.

What's $N_X(p)$?

The book answers this question in its first four lines. Paraphrasing slightly: Let X denote a set of multivariate polynomial equations with integer coefficients. Given a prime number p, we reduce all the polynomials modulo p and we count the number $N_X(p)$ of common solutions to the resulting congruences.

A simple case

Consider the equation

$$x + y = 0. \tag{1}$$

Picking the prime $p = 5$ gives rise to the congruence

$$x + y \equiv 0 \pmod 5,$$

which has the disappointingly easy solution $y \equiv -x \pmod 5$, leading us to the list

$$\{(0,0),(1,4),(2,3),(3,2),(4,1)\}.$$

Therefore $N_X(5) = 5$. More generally $N_X(p) = p$. Even more generally, replacing the field $\mathbb{F}_p$ with the field $\mathbb{F}_{p^n}$ for some $n \in \mathbb{Z}_{\geq 1}$, we get $N_X(p^n) = p^n$.

Not very interesting, you say?

Let's package the numbers $\{N_X(p^n) \mid n \in \mathbb{Z}_{\geq 1}\}$ into a function

$$\zeta_{X,p}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{N_X(p^n)}{n} \frac{1}{p^{ns}}\right) = \frac{1}{1-p^{s-1}}$$

and then throw all the prime numbers together[1]

$$\zeta_X(s) = \prod_{p \text{ prime}} \zeta_{X,p}(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{s-1}}.$$

Then $\zeta_X(s) = \zeta(s-1)$, where ζ is the Riemann zeta function. And suddenly, one of the simplest examples reveals some unexpected depth, and a relation with the most famous currently open problem in mathematics.

Another million-dollar problem

Of course, Serre is interested in equations more general than (1). The number $N_X(p)$ can be defined when X is the solution set of any system of integer polynomials. This is the bread and butter of (arithmetic) algebraic geometry, where such X is treated as a geometric object called a scheme over $\mathbb{Z}$. The example in (1) is a line, but more interesting geometry is just around the corner.

Let X be the set of solutions of the equation

$$y^2 - y = x^3 - x^2. \tag{2}$$

The numbers $N_X(p)$ turn out to be given by the formula[2]

$$N_X(p) = p + 1 - a_p,$$

where the integers a_n are the coefficients of the power series

$$F(q) = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2 = \sum_{n=1}^{\infty} a_n q^n.$$

This is in itself something magical: it says that the geometric object X (an elliptic curve) is related to the analytic object F (a modular form) in a way that illustrates

[1] Here s is a complex variable and the expressions may make sense only after restricting s to an appropriate half-plane, namely $\mathrm{Re}(s) > \dim(X)$.

[2] Well, almost. The formula holds if we view X as a curve in the projective plane $\mathbb{P}^2$ (rather than the affine plane). This amounts to turning (2) into its homogeneous version $y^2z - yz^2 = x^3 - x^2z$, discarding the solution $(0,0,0)$, and identifying solutions of the form (x,y,z) and $(\lambda x, \lambda y, \lambda z)$ for $\lambda \neq 0$.

a special case of the modularity conjecture of Shimura–Taniyama–Weil (the most famous consequence of which is Fermat's Last Theorem).

The zeta function of X, defined as before, turns out to be

$$\zeta_X(s) = \frac{\zeta(s)\zeta(s-1)}{L(s)},$$

where

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

And the complex function $L(s)$ has analytic continuation to all $s \in \mathbb{C}$, as predicted by the Birch and Swinnerton-Dyer conjecture. So we are only on page 11 of Serre's book and have already encountered two million-dollar problems and some of the most exciting concepts of the last half-century of number theory. As hard as it may be to believe, the remaining 150 pages live up to this promising start, with the Sato-Tate conjecture and the higher-dimensional prime number theorem making extended appearances.

The master plan I

The first aim of the book is to explore a central question about the numbers $N_X(p)$:

Fixing X, how does $N_X(p)$ vary with p?

Here is one of the answers given by Serre: Let $X(\mathbb{C})$ denote the complex variety defined by the equations of X. Then the dimension of $X(\mathbb{C})$ is less than d if and only if $N_X(p) = O(p^d)$ as the prime p gets large.

It is striking to notice the magical and easily stated way in which the topology of the complex variety $X(\mathbb{C})$ knows about the number of modulo p solutions of the original equations (and vice-versa). This is a leitmotif going back at least as far as the 1940s with André Weil's formulation of his famous conjectures, followed by the creation of several types of cohomology theories of arithmetic interest (and Deligne's proof of the Weil conjectures in the 1970s), and leading to the building of the still-largely-conjectural edifice of motives.

In addition to the issue of the variation of $N_X(p)$, the book touches upon several related questions. Efficient ways of computing the number of solutions are mentioned for several special types of X. Congruence properties modulo primes $\ell \neq p$ are used to show that if $N_X(p) = N_Y(p)$ for almost all primes p, then $N_X(p^n) = N_Y(p^n)$ for almost all primes p and all exponents n. Equidistribution results related to the numbers of solutions are proved to follow from a very general form of the Sato–Tate conjecture (special cases of which have recently been settled).

The master plan II

Serre declares upfront his second objective for the book: to review and bring together ideas from algebraic geometry, representations of profinite groups, several flavours of cohomology, algebraic and analytic number theory, and modular forms.

These topics have gone through significant developments over the last half-century, with Serre himself playing an important role in them.

The pedagogical aim of the text is supported by many beautiful examples, abundant and precise bibliographical notes, and by about 60 exercises (nontrivial, but with hints). Proofs start appearing after the four background chapters, and give just the right amount of detail to illuminate the argument without drowning it in technicalities.

Summary

The study of the number of solutions to equations is a particularly well suited pretext for this succinct, elegant, and readable introduction to modern mathematics. The book will be useful to advanced students and researchers who will benefit from a well organised exposition of the main issues surrounding the numbers $N_X(p)$ and related topics.

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