Puzzle Corner

Ivan Guo*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 37. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, Federation University Australia, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 37 is 1 July 2014. The solutions to Puzzle Corner 37 will appear in Puzzle Corner 39 in the September 2014 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Interesting intersection

Peter has drawn several (not necessarily convex) polygons on a piece of paper. He notices that any pair of the polygons have a non-empty intersection. Prove that Peter can draw a straight line which intersects all of the existing polygons.

Simplifying series

Simplify the following expression:

$$\frac{1}{1^4+1^2+1}+\frac{2}{2^4+2^2+1}+\cdots+\frac{100}{100^4+100^2+1}.$$

*School of Mathematics & Applied Statistics, University of Wollongong, NSW 2522, Australia.
Email: ivanguo1986@gmail.com

Cake cutting

Christie is holding a dinner party. It is known that either X or Y guests will attend. In preparation, Christie would like to cut a cake into some number of pieces (not necessarily of equal size), so that the cake can be equally shared between the guests in either scenario.

(i) If X and Y are relatively prime, what is the minimal number of pieces required to achieve this?

(ii) What if X and Y are not relatively prime?

Photo: Aneta Blaszczyk, SXC

Baffling bisection

Given an angle $\angle ABC$, it is well-known that we can construct its angle bisector using only a compass and a straight edge. For the reader's interest, the steps are as follows.

- Draw a circle centred at B, let it intersect the segments AB and AC at points P and Q respectively.
- Draw two (fairly large) circles of equal radii with centres at P and Q. Let these two circles intersect at points X and Y.
- The line XY is the required angle bisector.

In particular, note that the point B was used in the first step. Is it still possible to construct the angle bisector of $\angle ABC$ if the point B is not allowed to be used at all?

Solutions to Puzzle Corner 35

Many thanks to everyone who submitted. The \$50 book voucher for the best submission to Puzzle Corner 35 is awarded to Dave Johnson. Congratulations!

Row of reciprocals

Harry writes down a strictly increasing sequence of one hundred positive integers. He then writes down the reciprocals of the integers.

(i) *Is it possible for the sequence of reciprocals to form an arithmetic progression?*

(ii) *Apart from the last two reciprocals, is it possible for each reciprocal to be the sum of the next two?*

(iii) *Would the answers to the previous questions change if Harry had started with an infinite sequence instead?*

Solution by David Angell: (i) The answer is yes. Let $a_1, a_2, \ldots, a_{100}$ be any strictly decreasing arithmetic progression of positive integers. If Harry writes down

$$\frac{\prod_{i=1}^{100} a_i}{a_1},\ \frac{\prod_{i=1}^{100} a_i}{a_2},\ \ldots\ \frac{\prod_{i=1}^{100} a_i}{a_{99}},\ \frac{\prod_{i=1}^{100} a_i}{a_{100}},$$

which is a strictly increasing sequence of positive integers, then the reciprocals form an arithmetic progression.

(ii) Again the answer is yes. Apply the same construction to a decreasing 'reverse Fibonacci' sequence of positive integers $a_1, a_2, \ldots, a_{100}$ with $a_{99} > a_{100}$ and

$$a_{98} = a_{99} + a_{100}, \quad a_{97} = a_{98} + a_{99}, \quad \ldots, \quad a_1 = a_2 + a_3.$$

Once again, the sequence

$$\frac{\prod_{i=1}^{100} a_i}{a_1},\ \frac{\prod_{i=1}^{100} a_i}{a_2},\ \ldots,\ \frac{\prod_{i=1}^{100} a_i}{a_{99}},\ \frac{\prod_{i=1}^{100} a_i}{a_{100}}$$

satisfies the required properties.

(iii) Neither construction is now possible. For (i), any strictly decreasing arithmetic progression must eventually be negative. But since the reciprocals are always positive rationals, this is clearly impossible.

For (ii), suppose that we have an infinite sequence of positive integers $a_1, a_2, \ldots$ with the property that

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_3}, \quad \frac{1}{a_2} = \frac{1}{a_3} + \frac{1}{a_4}$$

and so on. It is easily shown by induction that, for $n \geq 3$,

$$\frac{(-1)^n}{a_n} = \frac{f_{n-1}}{a_2} - \frac{f_{n-2}}{a_1},$$

where the f_ns are the Fibonacci numbers beginning with $f_1 = f_2 = 1$. We may rewrite this as

$$\frac{(-1)^n a_2}{a_n f_{n-2}} = \left(\frac{f_{n-1}}{f_{n-2}} - \frac{a_2}{a_1}\right).$$

Since the left-hand side converges to 0 as $n \to \infty$, we must have a_2/a_1 converging to the golden ratio ϕ,

$$\frac{a_2}{a_1} = \lim_{n\to\infty} \frac{f_{n-1}}{f_{n-2}} = \phi = \frac{1+\sqrt{5}}{2}.$$

This is impossible since a_2/a_1 is rational but ϕ is irrational.

Pebble placement

(i) *There are several pebbles placed on an $n \times n$ chessboard, such that each pebble is inside a square and no two pebbles share the same square. Perry decides to play the following game. At each turn, he moves one of the pebbles to an empty neighbouring square. After a while, Perry notices that every pebble has passed through every square of the chessboard exactly once and has come back to its original position.*

Prove that there was a moment when no pebble was on its original position.

(ii) *Peggy aims to place pebbles on an $n \times n$ chessboard in the following way. She must place each pebble at the centre of a square and no two pebbles can be in the same square. To keep it interesting, Peggy makes sure that no four pebbles form a non-degenerate parallelogram.*

What is the maximum number of pebbles Peggy can place on the chessboard?

Solution by Dave Johnson: (i) Consider the moment immediately after the move that takes the last pebble out of its original square, say S. At this moment, all the other pebbles have left their original squares. Also none of them has passed through S, so they cannot yet return to their original positions. Thus at this time, no pebble was on its original position, as required.

(ii) The maximum number is $2n-1$. This may be achieved by placing pebbles all along the top row and down the leftmost column. It is clear that in this configuration, no four pebbles may form a parallelogram.

Now we show that it is not possible to place $2n$ or more pebbles without creating parallelograms. Begin with any configuration of at least $2n$ pebbles. Suppose that there are r_i pebbles in row i. In that row, the distances between the leftmost pebble and the other $r_i - 1$ pebbles are all distinct, forming $r_i - 1$ distances. Summing over all the rows, there must be at least

$$\sum_{i=1}^{n}(r_i - 1) \geq 2n - n = n$$

distances from the leftmost pebbles. Since the possible distances must take values from $1, 2, \ldots, n-1$, there must be two pairs of pebbles from two distinct rows with equal horizontal distances, creating a parallelogram.

Flawless harmony

Call a nine-digit number flawless *if it has all the digits from 1 to 9 in some order. An unordered pair of flawless numbers is called* harmonious *if they sum to 987654321. Note that* (a, b) *and* (b, a) *are considered to be the same unordered pair.*

Without resorting to an exhaustive search, prove that the number of harmonious pairs is odd.

Solution by Joe Kupka: As an example, one harmonious pair is given by

$$(123456789,\ 864197532).$$

If we switch the order of the last two digits, the new pair

$$(123456798,\ 864197523)$$

also happens to harmonious. This is not a coincidence. In general, suppose we have a pair (m, n) with

$$m = 100X + 10a + b, \quad n = 100Y + 10c + d,$$

where $2 \le a, b, c, d \le 9$. In order for (m, n) to be harmonious, or $m+n = 987654321$, we must have $a + c = b + d = 11$. Then it is clear that the pair (m', n') given by

$$m' = 100X + 10b + a, \quad n' = 100Y + 10d + c$$

also satisfies $m' + n' = 987654321$. Thus (m', n') is harmonious as well.

In most cases, the pairs (m, n) and (m', n') are different. The notable exception is when we have $m = n'$ and $n = m'$. This can only happen if $A = C$, $a = d$ and $b = c$. It is easy to check (by working out the digits of $A = C$ backwards) that the only pair satisfying these equalities is given by

$$(493827156,\ 493827165).$$

Therefore the number of harmonious pairs must be odd.

Balancing act

There are some weights on the two sides of a balance scale. The mass of each weight is an integer number of grams, but no two weights on the same side of the scale share the same mass. At the moment, the scale is perfectly balanced, with each side weighing a total of W *grams. Suppose* W *is less than the number of weights on the left multiplied by the number of weights on the right.*

Is it always true that we can remove some, but not all, of the weights from each side and still keep the two sides balanced?

Solution by Jensen Lai: Let the number of weights on each side of the scale be x and y, so $W < xy$. If $x = 1$, then $W < y$. This is not possible since each of the y weights have integral masses. Thus $x > 1$ and similarly, $y > 1$.

Assume now that no mass appears on both sides of the scale. Since no two weights on the same side share the same mass, all $x+y$ weights are unique. Since they all have integral masses, the minimum total mass of the weights is $(x+y)(x+y+1)/2$. This gives

$$\begin{aligned} & \frac{(x+y)(x+y+1)}{2} \leq 2W < 2xy \\ \implies \quad & (x+y)(x+y+1) < 4xy \\ \implies \quad & (x-y)^2 + x + y < 0. \end{aligned}$$

Since x and y are positive, this implies $(x-y)^2 < 0$ which is impossible.

Thus there must exist a mass, say M, which appears on both sides of the scale. We can then simply remove M and keep the two sides balanced. Note that since $x, y > 1$, we have not removed all the weights.

Ivan is a Postdoctoral Research Fellow in the School of Mathematics and Applied Statistics at The University of Wollongong. His current research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.