Puzzle Corner

Ivan Guo*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 36. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, Federation University Australia, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 36 is 1 May 2014. The solutions to Puzzle Corner 36 will appear in Puzzle Corner 38 in the July 2014 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

World cup

In the soccer world cup, each group has four teams. Each team plays one game with every other team in its group. A win gives 3 points, a draw 1 point and a loss 0 points. From each group, two teams advance so that each advancing team gets at least as many points as each non-advancing team.

(i) What is the smallest possible score of an advancing team?

(ii) What is the largest possible score of a non-advancing team?

*School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia.
Email: ivanguo1986@gmail.com

Polynomial product

Let n be a positive integer. Consider the polynomial:

$$P(x) = (1+x)(2+x^2)(3+x^4)(4+x^8)\cdots(n+x^{2^{n-1}}).$$

Express the product of the non-zero coefficients of $P(x)$ in terms of n.

Coin coverage

One hundred identical coins lie on a rectangular table, in such a way that no more can be added without overlapping. We allow a coin to extend over the edge of the table, as long as its centre is still on the table.

Prove that if overlapping is allowed, it is possible to start again and completely cover the table with four hundred of these coins.

Matching remainders

Submitted by Norman Do

The numbers $1, 2, \ldots, 2n$ are divided into two groups of n numbers. We form a list of the remainders formed by dividing the sums $a+b$ by $2n$, where a, b are in the same group (and may be equal).

Prove that the n^2 remainders from one group are equal, in some order, to the n^2 remainders of the other group.

Tessellation test 2

Tess is tessellating polygons with parallelograms again. This time she has successfully divided a regular $4n$-gon into various parallelograms.

(i) Prove that at least n of the parallelograms are rectangles.

(ii) If the original regular $4n$-gon has unit side lengths, prove that the sum of the areas of all rectangles equals to n.

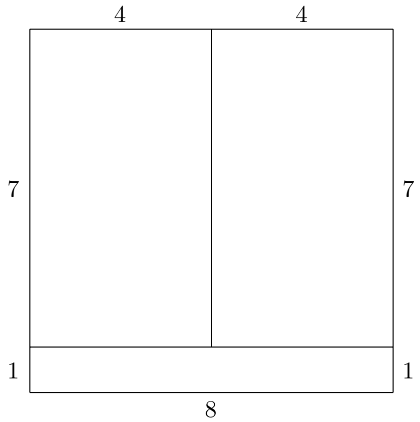
Solutions to Puzzle Corner 34

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 34 is awarded to Joe Kupka. Congratulations!

Crowded square

There are four points inside an 8 metres by 8 metres square. Prove that two of those points are at most $\sqrt{65}$ metres apart.

Solution by Peter McNamara: Divide the square up into three rectangles as shown in the following diagram.



Since $7^2 + 4^2 = 8^2 + 1^2 = 65$, each rectangle has a diagonal length of $\sqrt{65}$ metres. By the pigeon-hole principle, two of the points must lie in the same rectangle. Hence the distance between those two points cannot be more than $\sqrt{65}$ metres.

Note: The best bound possible is actually 8 metres, but the choice of $\sqrt{65}$ allows for the nice solution as presented above.

Fraction practice 2

Franny is practising her fractions again. She begins with the numbers

$$\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{100}$$

written on the board. At each turn, Franny may erase two numbers a, b and replace them with a single number $f(a,b)$. This is repeated until only one number remains.

(i) *If $f(a,b) = ab/(a+b)$, what are the possible values of the final number?*

(ii) *If $f(a,b) = ab + a + b$, what are the possible values of the final number?*

Solution by Martin Bunder: (i) Whenever the fractions $\frac{1}{m}$ and $\frac{1}{n}$ are erased, they are replaced by

$$f\left(\frac{1}{m}, \frac{1}{n}\right) = \frac{\frac{1}{mn}}{\frac{1}{m} + \frac{1}{n}} = \frac{1}{m+n}.$$

So the sum of the reciprocals of the fractions on the board is constant. Thus the final fraction must be

$$\frac{1}{1 + 2 + \cdots + 100} = \frac{1}{5050}.$$

(ii) First note that

$$f(a,b) + 1 = ab + a + b + 1 = (a+1)(b+1).$$

So if we add 1 to each of the numbers on the board and then multiply everything together, the product is a constant. Thus the final number must be

$$\left(\frac{1}{1}+1\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{100}+1\right)-1=\frac{2}{1}\times\frac{3}{2}\times\cdots\times\frac{101}{100}-1=100.$$

Prickly pair

I am thinking of a pair of positive integers. To help you work out what they are, I will give you some clues. Their difference is a prime, their product is a perfect square, and the last digit of their sum is 3. What can they possibly be?

Solution by Dave Johnson: Let the pair of numbers be x and y with $x < y$, and denote their difference $y - x$ by the prime number p. Since $\gcd(x, y) \mid p$, there are two possible cases.

If $\gcd(x, y) = p = y - x$, then write $x = pa$ and $y = p(a + 1)$. Since the product $xy = p^2a(a + 1)$ is a square, the number $a(a + 1)$ must be a square. But this is impossible since $a^2 < a(a + 1) < (a + 1)^2$.

If $\gcd(x, y) = 1$, then x and y are coprime. The product xy being a square implies that both x and y are squares. Writing $x = a^2$ and $y = b^2$, we have

$$p = b^2 - a^2 = (b - a)(b + a),$$

which implies $b = a + 1$ and $p = a + b = 2a + 1$. Now using the fact that the last digit of the sum is 3, we must have

$$x + y = a^2 + b^2 = 2a^2 + 2a + 1 = 10k + 3$$

for some integer k. This simplifies to $a^2 + a = 5k + 1$. It is easy to check that in modulo 5, the only possible solution is $a \equiv 2$, which implies that $p = 2a + 1 \equiv 0$. Thus $p = 5$, and the only two square numbers which differ by 5 are 4 and 9.

Therefore the required pair of integers is $(4, 9)$.

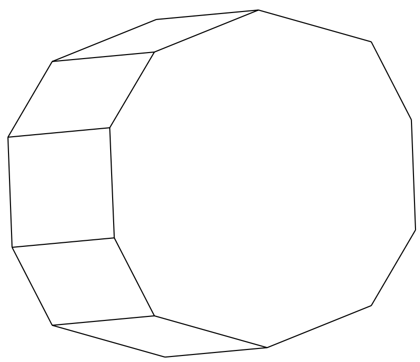
Tessellation test

Tess is trying to draw an n-sided convex polygon which can be tessellated by a finite number of parallelograms. For which n will Tess be able to succeed?

Solution by Jensen Lai: We claim that it is possible to tessellate an n-gon by parallelograms if and only if n is even.

Consider a side S of the n-gon and suppose it is on the bottom. Any parallelogram touching S must have a top edge E_1 which is parallel to S. Then we may find a parallelogram that has E_1 as the bottom edge. Its top edge E_2 is still parallel to S. This process can be repeated until we reach an edge E_k that is on the boundary of the original n-gon, belonging to some side S'. Hence for any side S of the n-gon, there exists another side S' that is parallel to S. Since in a convex polygon, any side can be parallel to at most one other side, we can hence group the sides of our n-gon into parallel pairs. This automatically implies that n is even.

Now we shall show that such an n-gon exists for even n by induction. For the base case of $n = 4$, we can simply take any parallelogram. Now suppose the tessellation is possible for a $2k$-sided polygon P_{2k}. We can construct a $(2k+2)$-sided polygon P_{2k+2} by translating k sides of P_{2k} while creating k additional parallelograms. The following diagram demonstrates the construction that turns a 10-gon into a 12-gon, but the same idea works for any even number.



Diminishing differences

Begin with n integers $x_1, \ldots, x_n$ around a circle. At each turn, simultaneously replace all of them by the absolute differences

$$|x_1 - x_2|,\ |x_2 - x_3|,\ \ldots,\ |x_{n-1} - x_n|,\ |x_n - x_1|.$$

Repeat this process until every number is 0, then stop. Prove that this process always terminates if and only if n is a power of 2.

Solution by Joe Kupka: We shall denote the initial sequence by

$$s = (x_1, x_2, \ldots, x_n)$$

and the operation by the function f, or

$$f((x_1, x_2, \ldots, x_n)) = (|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_{n-1} - x_n|, |x_n - x_1|).$$

The sequence reached after k operations is denoted by $f^k(s)$. Throughout the solution, the indices will be taken in modulo n. Since the terms will be non-negative after the first step, we shall assume that only non-negative integers are involved. Furthermore, it is clear that the numbers appearing in these sequences are bounded.

First suppose that $n > 1$ is odd. Consider the starting sequence $s = (1, 0, \ldots, 0)$. It is clear that we will only ever have 1s and 0s. If we eventually have all 0s, let the first occurrence of this be $f^k(s) = (0, 0, \ldots, 0)$. Then in the step before, we must have $f^{k-1}(s) = (1, 1, \ldots, 1)$. Going back further, the numbers in $f^{k-2}(s)$ must alternate between 1s and 0s. But this is not possible since n is odd.

Now suppose that n has an odd factor $m > 1$. Consider the starting sequence

$$s = (\underbrace{1, 0, \ldots, 0}_{m}, \underbrace{1, 0, \ldots, 0}_{m}, \ldots, \underbrace{1, 0, \ldots, 0}_{m}).$$

Due to the cyclic nature of the process, the sequence $f^i(s)$ will always have period m. Furthermore, each chunk of length m has the same behaviour as an evolving length m sequence starting with $(1, 0, \ldots, 0)$. Thus by previous arguments, it is not possible to reach all 0s if n has an odd factor.

Finally, we show that it is always possible to reach all 0s if $n = 2^a$ where a is a positive integer. Consider everything in modulo 2. Since $x - y \equiv x + y$, we may replace the differences by sums. It is possible to explicitly compute $f^n(s)$ in modulo 2. Begin by computing $f^2(s)$:

$$\begin{aligned} s &= (x_1, x_2, \ldots, x_n), \\ f(s) &\equiv (x_1 + x_2, x_2 + x_3, \ldots, x_n + x_1), \\ f^2(s) &\equiv (x_1 + 2x_2 + x_3, x_2 + 2x_3 + x_4, \ldots, x_n + 2x_1 + x_2) \\ &\equiv (x_1 + x_3, x_2 + x_4, \ldots, x_n + x_2). \end{aligned}$$

Then $f^4(s)$ can be computed by applying $f^2(s)$ twice:

$$\begin{aligned} f^4(s) &\equiv f^2(f^2(s)) \\ &\equiv (x_1 + 2x_3 + x_5, x_2 + 2x_4 + x_6, \ldots, x_n + 2x_2 + x_4) \\ &\equiv (x_1 + x_5, x_2 + x_6, \ldots, x_n + x_4). \end{aligned}$$

Continuing this pattern, we can compute $f^{2^i}(s)$ for any i. In particular, for $f^n(s)$,

$$\begin{aligned} f^n(s) &= f^{2^a}(s) \\ &\equiv (x_1 + x_{2^a+1}, x_2 + x_{2^a+2}, \ldots, x_n + x_{2^a}) \\ &\equiv (0, 0, \ldots, 0). \end{aligned}$$

Since this is in modulo 2, we have shown that, after $n = 2^a$ iterations, all numbers must be even. By dividing everything by 2 and applying the same process, we see that after a further n iterations, all numbers must be multiples of 4. Repeating this argument, we see that after kn iterations, all numbers must be multiples of 2^k, for all k. But since the numbers are bounded, we must eventually have all 0s. This completes the solution.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.