



Technical Papers

Lift-Off Fellowship report On the periodicity of subtraction games

Nhan Bao Ho*

My PhD thesis is aligned with the field of combinatorial games. A combinatorial game is a two-player game in which the players move alternately with neither skipping nor cooperation. There is no hidden information and no element of luck, and so each player understands well the state of the game after each move. There is typically a finite number of moves, so the game must end. The player who makes the last move wins.

In my PhD research (under the supervision of Dr Grant Cairns at La Trobe University), I investigated several variants of the game known as Nim. Many questions had become apparent during this work and the Lift-off Fellowship gave me an excellent chance to expand my study.

My first task was to examine my conjecture on the periodicity of subtraction games. A subtraction game involves a pile of tokens and a finite set S . Two players take turns to move, removing a number of tokens from the given pile, provided that this number belongs to S . It is well known that subtraction games are ultimately periodic [4]. However, there has not been a comprehensive solution for the periodicity of subtraction games, even with 3-element subtraction set S .

In my early work [2], I conjectured from the computation that the periodicity of the nim-sequence of a subtraction game and that of the winning/losing state coincide. The conjecture proposes a link to the periodicity of octal games in which subtraction games form a special subclass. Many attempts were made on this conjecture. I also gave a talk on this conjecture at SIAM Conference on Discrete Mathematics (Dalhousie University, June 2012). The conjecture was disproved recently, based on counterexamples found by Grant Cairns. For example, in the subtraction game with $S = \{4, 6, 11, 14\}$, the period of winning/losing state is 17 while the period of nim-sequence is 34. A further draft of this article was submitted for publication.

With the support of the fellowship, I was able to expand my work on another game called Max-Welter. The paper was finally published in *Discrete Mathematics* [3].

During this fellowship, I also obtained some progress on solving a problem on Wythoff's game I proposed in my thesis. This resulted in an article [1], written

*Department of Mathematics, La Trobe University, Melbourne, Australia 3086.
Email: nhan.ho@latrobe.edu.au, nhanbaoho@gmail.com

jointly with Professor Aviezri S. Fraenkel. This work was presented in Integers Conference 2013 (University of West Georgia) and will be submitted for publication.

I gratefully acknowledge the support of the Lift-off Fellowship. It provided me a perfect boost after my PhD research.

References

- [1] Fraenkel, A.S. and Ho N.B. When are translations of P-positions of Wythoff's game P-positions? Preprint.
- [2] Ho, N.B. Subtraction games with three element subtraction sets. Preprint. Available at <http://arxiv.org/abs/1202.2986v1>.
- [3] Ho, N.B. The Max–Welter game. (2014). *Discrete Math.* **318**, 41–47.
- [4] Guy, R.K. (1991). *Fair game: How to Play Impartial Combinatorial Games*. COMAP, Arlington, MA.



I completed my Bachelor of Science (Mathematics) in 2002 in Vietnam. From 2006 to 2007, I studied Master of Science and Technology Education at La Trobe University, with an honour thesis titled ‘Completely positive maps for real and complex C^* -algebras’, under the supervision of Associate Professor Peter Stacey. I obtained my PhD degree in 2012, with a thesis titled ‘Combinatorial aspects of variants of the game of Nim’, under the supervision of Associate Professor Grant Cairns. I will finish my six-month Endeavor Research Fellowship at Monash University in August 2014 before returning to my home country to continue my teaching career at Quangtri Teacher Training College.