Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 38. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, Federation University Australia, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 38 is 1 September 2014. The solutions to Puzzle Corner 38 will appear in Puzzle Corner 40 in the November 2014 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Surface temperature

For the purpose of this puzzle, let us assume that the Earth is perfectly spherical, and the surface temperature is a continuous function of the Earth's surface.

(i) Prove that there exist two antipodal points with the same surface temperature.

(ii) Fix a distance d less than the diameter of the Earth. Prove that there exist two points exactly d apart, that have the same surface temperature.

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Triangle existence

(i) For which integer values of x does there exist a non-degenerate triangle with side lengths of 5, 10 and x?

(ii) In a triangle, an altitude length refers to the perpendicular distance from a vertex to the opposite side. For which integer values of x does there exist a non-degenerate triangle with altitude lengths of 5, 10 and x?

Colourful lattice

In the coordinate plane, points with integer coordinates are called *lattice points*.

(i) Suppose that each lattice point is coloured using one of n possible colours. Prove that there exist four lattice points with the same colour which are also the vertices of a rectangle.

(ii) Suppose that each lattice point is either coloured using one of n possible colours, or not coloured at all. Furthermore, suppose that it is not possible to find four lattice points with the same colour which are also the vertices of a rectangle. Prove that there exist arbitrarily large squares such that none of lattice points in their interior is coloured at all.

Drawing parallels

Two parallel lines are drawn on a sheet of paper. There is also a marked point which does not lie on either of these lines. Here is your challenge: using only an unmarked straight edge (and no compass), construct a new line through the marked point, that is also parallel to the two existing lines.

Bonus: Can you find two different ways to achieve this?

Solutions to Puzzle Corner 36

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 36 is awarded to Jensen Lai. Congratulations!

World cup

In the soccer world cup, each group has four teams. Each team plays one game with every other team in its group. A win gives 3 points, a draw 1 point and a loss 0 points. From each group, two teams advance so that each advancing team gets at least as many points as each non-advancing team.

(i) *What is the smallest possible score of an advancing team?*

(ii) *What is the largest possible score of a non-advancing team?*

Solution by Alan Jones:

(i) The answer is 2. A team T which scores 0 or 1 points has lost at least twice. Therefore there are at least two others teams with 3 or more points, so T cannot advance.

However, if one team has three victories and the other three teams draw all of their other matches, there are three teams with 2 points, one of which must advance.

(ii) The answer is 6. A team T which scores 7 or more has won at least twice. Therefore there are at least two other teams with 6 or less, so T must advance.

However, if A beats B, B beats C, C beats A, and everyone beats D, then A, B and C all have 6 points and one of them does not advance.

Polynomial product

Let n be a positive integer. Consider the polynomial:

$$P(x) = (1 + x)(2 + x^2)(3 + x^4)(4 + x^8) \cdots (n + x^{2^{n-1}}).$$

Express the product of the non-zero coefficients of $P(x)$ in terms of n.

Solution by Aaron Hassan: We first note that when $P(x)$ is expanded, there are exactly 2^n non-zero terms, in the following form:

$$P(x) = a_0x^0 + a_1x^1 + \cdots + a_{2^n-1}x^{2^n-1}.$$

This follows from the fact that each non-negative integer has a unique binary expansion, so each term x^i can be uniquely written as the product of a subset of $\{x^1, x^2, x^4, \ldots, x^{2^{n-1}}\}$. Hence the n brackets will produce 2^n terms with no like terms to be collected.

Now the 2^n coefficients of $P(x)$ are simply the products of the 2^n possible subsets of $\{1, 2, 3, \ldots, n\}$. To compute the overall product of the coefficients, it suffices to compute the number of times each element from $\{1, 2, 3, \ldots, n\}$ appears in the final expansion. In particular, when the bracket $(i + x^{2^i-1})$ is expanded, i will be used half of the time while x^{2^i-1} will be used during the other half. This implies that each i will appear in $2^n/2 = 2^{n-1}$ coefficients.

Therefore the required product of the non-zero coefficients must be $(n!)^{2^{n-1}}$.

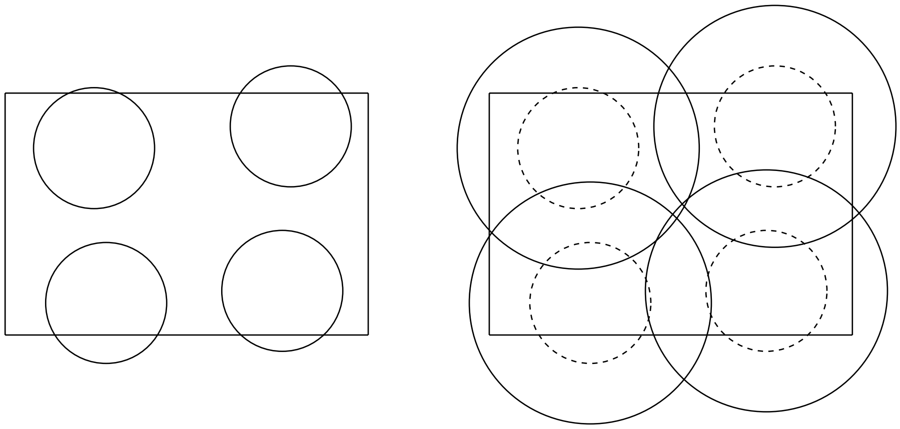
Coin coverage

One hundred identical coins lie on a rectangular table, in such a way that no more can be added without overlapping. We allow a coin to extend over the edge of the table, as long as its centre is still on the table.

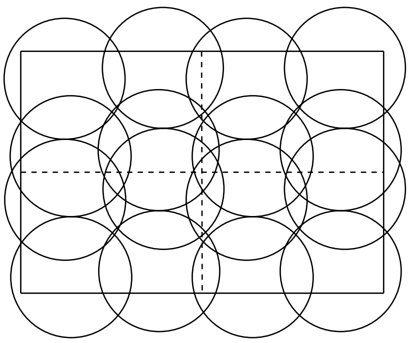
Prove that if overlapping is allowed, it is possible to start again and completely cover the table with four hundred of these coins.

Solution: Without loss of generality, let the radius of each coin be 1 cm. Initially, there are 100 coins with 100 *coin centres*. We claim that every point on the table is within 2cm of a coin centre. Suppose it is not the case at some point P. Then it is possible to place a new coin with centre P without creating any overlaps, since every other coin is more than 1cm from P. This is a contradiction.

Now let us replace the 100 coins by 100 over-sized coins with 2cm radii, but retaining the same 100 coin centres. Since no point on the table is more than 2cm from a coin centre, the new over-sized coins must cover the table completely.



With a little bit of magic, let us shrink the table as well as the coins by a length factor of 2. Now we have 100 normal-sized coins completely covering a small-sized table, which is half as long and half as wide as the original. Since a normal-sized table can be divided into four quarters, each identical to a small-sized table, we can complete the trick by simply repeating the same coin arrangement for each of the four quarters. As a result, we have managed to cover the normal-sized table with 400 normal-sized coins, as required.



Matching remainders

The numbers $1, 2, \ldots, 2n$ are divided into two groups of n numbers. We form a list of the remainders formed by dividing the sums $a + b$ by $2n$, where a, b are in the same group (and may be equal).

Prove that the n^2 remainders from one group are equal, in some order, to the n^2 remainders of the other group.

Solution by Joe Kupka: Let the two sets of size n be S_1 and S_2. Fix a particular remainder $r \in \{0, 1, \ldots, 2n-1\}$. We shall proceed by counting the number of of times r appears when summing within S_1 and within S_2.

We work under modulo $2n$. Consider the list of all possible ordered pairs from $\{1, 2, \ldots, 2n\}$ which sum to r, they are:

$$P_1 = (1, r-1),\ P_2 = (2, r-2),\ \ldots,\ P_{2n-1} = (2n-1, r+1),\ P_{2n} = (2n, r).$$

Note that each element of $\{1, 2, \ldots, 2n\}$ appears exactly twice in the list. Since $|S_1| = n$, the elements of S_1 collectively appear $2n$ times in the list. The same holds for S_2.

Now define $f(P_i)$ to be the number of elements in P_i that is from S_1. So for each i, $f(P_i) = 0, 1$ or 2. But on average,

$$\frac{1}{2n}\sum_{i=1}^{2n} f(P_i) = \frac{1}{2n} 2n = 1.$$

Thus the number of occurrences of $f(P_i) = 0$ must equal the number of occurrences of $f(P_i) = 2$. In other words, the number of pairs with both elements in S_1 is equal to the number of pairs with both elements in S_2.

Therefore, the remainder r must occur the same number of times in S_1 and S_2. Since this is true for any remainder r, the n^2 remainders of sums in S_1 must be equal, in some order, to the n^2 remainders of sums in S_2.

Ivan is a Postdoctoral Research Fellow in the School of Mathematics and Applied Statistics at The University of Wollongong. His current research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.