Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 34. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of \$50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, University of Ballarat, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 34 is 1 November 2013. The solutions to Puzzle Corner 34 will appear in Puzzle Corner 36 in the March 2014 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Crowded square

There are four points inside an 8 metres by 8 metres square. Prove that two of those points are at most $\sqrt{65}$ metres apart.

Fraction practice 2

Franny is practising her fractions again. She begins with the numbers

$$\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{100}$$

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This puzzle corner is also featured on the Mathematics of Planet Earth Australia website http://mathsofplanetearth.org.au/

written on the board. At each turn, Franny may erase two numbers a, b and replace them with a single number $f(a, b)$. This is repeated until only one number remains.

(i) If $f(a, b) = ab/(a + b)$, what are the possible values of the final number?

(ii) If $f(a, b) = ab + a + b$, what are the possible values of the final number?

Prickly pair

I am thinking of a pair of positive integers. To help you work out what they are, I will give you some clues. Their difference is a prime, their product is a perfect square, and the last digit of their sum is 3. What can they possibly be?

Tessellation test

Tess is trying to draw an n-sided convex polygon which can be tessellated by a finite number of parallelograms. For which n will Tess be able to succeed?

Diminishing differences

Begin with n integers $x_1, \ldots, x_n$ around a circle. At each turn, simultaneously replace all of them by the absolute differences

$$|x_1 - x_2|,\ |x_2 - x_3|,\ \ldots,\ |x_{n-1} - x_n|,\ |x_n - x_1|.$$

Repeat this process until every number is 0, then stop. Prove that this process always terminates if and only if n is a power of 2.

Solutions to Puzzle Corner 32

Many thanks to everyone who submitted. The \$50 book voucher for the best submission to Puzzle Corner 32 is awarded to Dave Johnson. Congratulations!

Telescoping product

Let n be an integer greater than 1. Simplify

$$\frac{2^3 - 1}{2^3 + 1} \times \frac{3^3 - 1}{3^3 + 1} \times \cdots \times \frac{n^3 - 1}{n^3 + 1}.$$

Solution by Jeremy Ottenstein: We begin by noting the following identity

$$\frac{k^3 - 1}{k^3 + 1} = \frac{(k-1)(k^2 + k + 1)}{(k+1)(k^2 - k + 1)} = \frac{(k-1)((k+1)^2 - k)}{(k+1)(k^2 - (k-1))}.$$

Applying to the current problem,

$$\begin{aligned}
&\frac{2^3-1}{2^3+1}\times\frac{3^3-1}{3^3+1}\times\cdots\times\frac{n^3-1}{n^3+1}\\
&\quad=\frac{1(3^2-2)}{3(2^2-1)}\times\frac{2(4^2-3)}{4(3^2-2)}\times\cdots\times\frac{(n-1)((n+1)^2-n)}{(n+1)(n^2-(n-1))}\\
&\quad=\frac{1\times 2\times\cdots\times(n-1)}{3\times 4\times\cdots\times(n+1)}\,\frac{(n+1)^2-n}{2^2-1}\\
&\quad=\frac{2}{3}\,\frac{n^2+n+1}{n^2+n}.
\end{aligned}$$

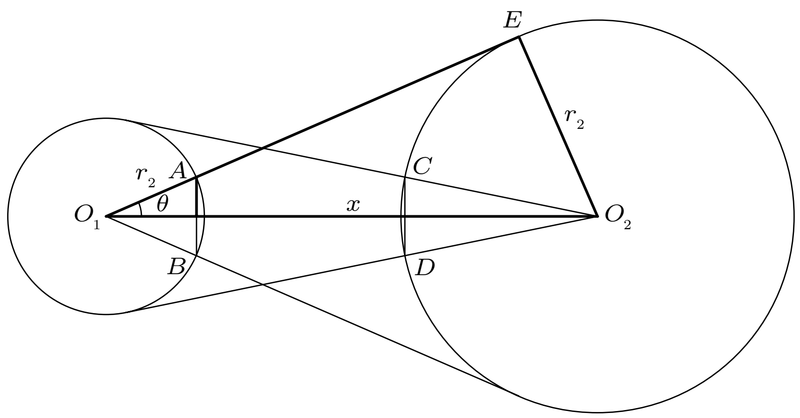
Tangent intersections

Let Γ_1 and Γ_2 be two non-overlapping circles with centres O_1 and O_2 respectively. From O_1, draw the two tangents to Γ_2 and let them intersect Γ_1 at points A and B.

Similarly, from O_2, draw the two tangents to Γ_1 and let them intersect Γ_2 points by C and D.

Prove that $AB = CD$.

Solution by Martin Bunder: Denote the radii of Γ_1 and Γ_2 by r_1 and r_2 respectively. Let the line O_1A be tangent to Γ_2 at E. Furthermore let $\angle O_2O_1E = \theta$ and $O_1O_2 = x$. Refer to the diagram below.



By considering the two right angled triangles involving θ, we have

$$AB = 2r_1 \sin\theta = 2r_1r_2/x.$$

Similarly, $CD = 2r_2r_1/x = AB$, as required.

Comment: Fittingly, this result is known as *the eyeball theorem*.

Colour coordination

Submitted by Joe Kupka

I need to hang 20 garments on a clothes line. Each garment requires two pegs. I have 20 green and 20 red pegs. I choose pegs at random. On average, how many garments will have pegs of the same colour?

Solution by Stephen Clarke: We will solve the generalised problem for n garments and n pegs of each colour.

Fix the first peg and suppose it is green. Now consider the possible choices for second peg. Out of the remaining $2n-1$ pegs, $n-1$ of them are green. Hence the probability of the first two pegs being green (knowing that the first one is green) is $(n-1)/(2n-1)$. The same argument can be made if the first peg is red. Therefore the probability of the first garment having matching pegs is $(n-1)/(2n-1)$.

Repeating the argument for each of the n garments, we see that the expected number of garments with matching pegs is $n(n-1)/(2n-1)$. In particular, when $n = 20$, the answer is $380/39$.

Team tactics 2

In a game show, there are three girls, each wearing a blue or a red hat. Each girl can only see the hats of the other two but not her own. Without any communication between themselves, each girl has to choose a real number and whisper it to the host. At the end, the host will add up the numbers chosen by girls wearing red hats, then subtract the numbers chosen by girls wearing blue hats. The girls win if the final answer is positive.

Before the show, the girls try to devise a strategy to maximise their probability of winning.

(i) *What is the maximum probability of winning?*

(ii) *If the girls were only allowed to choose from* $\{-1, 0, 1\}$, *what is the maximum probability of winning?*

Bonus: *If there are seven girls instead of three, and each girl can see the hats of the other six but not her own, how do the answers change?*

Solution: (i) There are $2^3 = 8$ possible scenarios. From the perspective of each girl, her own hat colour is independent of what she observes. So no matter what number she chooses, it has equal chance of being added or subtracted to the total score. Hence the expected total score is 0. In particular, regardless of the strategy chosen, there must be at least one scenario in which the girls lose.

We now construct a strategy which wins in 7 of the 8 scenarios. Since the girls wearing red hats will always see more blue hats than the girls wearing blue hats, it makes sense to choose larger numbers if more blue hats are visible. Suppose each girl chooses 3^x, where x is the number of blue hats visible. The possible outcomes are

- RRR: total score $= 1 + 1 + 1 = 3$.
- RRB, RBR, BRR: total score $= 3 + 3 - 1 = 5$.
- RBB, BRB, BBR: total score $= 9 - 3 - 3 = 3$.
- BBB: total score $= -9 - 9 - 9 = -27$.

So the girls win if there is at least one red hat. Therefore the maximum probability of winning is 7/8.

Bonus: For 7 girls, the same argument shows that the maximum probability of winning is 127/128. This can be achieved if each girl chooses 7^x, where x is the number of blue hats she sees.

(ii) Again the expected total score is 0. Suppose the girls win in k of the 8 scenarios. Since each winning scenario requires at least 1 point and each losing scenario can be no less than -3 points, we must have

$$k - 3(8 - k) \leq 0 \quad \Longrightarrow \quad k \leq 6.$$

We now construct a strategy which wins in 6 of the 8 scenarios. Suppose each girl chooses 1 if she sees 2 blue hats, -1 if she sees 2 red hats, and 0 otherwise. The possible outcomes are

- RRR: score $= -1 - 1 - 1 = -3$.
- RRB, RBR, BRR: score $= -0 - 0 - (-1) = 1$.
- RBB, BRB, BBR: score $= 1 - 0 - 0 = 1$.
- BBB: score $= -1 - 1 - 1 = -3$.

So the girls win unless all hats are of the same colour. This achieves the maximum winning probability of 6/8.

Bonus: For 7 girls, the same argument shows that the maximum probability of winning is 112/128. The construction achieving this is a little trickier. For brevity, we will simply present the 16 losing scenarios (1 and 0 are used instead of red and blue for clarity):

$$\begin{array}{llll} 1111111, & 0010111, & 1001011, & 1100101, \\ 1110010, & 0111001, & 1011100, & 0101110, \\ 0000000, & 1101000, & 0110100, & 0011010, \\ 0001101, & 1000110, & 0100011, & 1010001. \end{array}$$

The details of the strategy will be left to the readers.

A sequence of sequences

Let $S_1, S_2, \ldots$ be finite sequences of positive integers defined in the following way. Set $S_1 = (1)$. For $n > 1$, if $S_{n-1} = (x_1, \ldots, x_m)$ then

$$S_n = (1, 2, \ldots, x_1, 1, 2, \ldots, x_2, \ldots, 1, 2, \ldots, x_m, n).$$

For example, the next few sequences are $S_2 = (1, 2)$, $S_3 = (1, 1, 2, 3)$ and $S_4 = (1, 1, 1, 2, 1, 2, 3, 4)$.

Prove that in the sequence S_n where $n > 1$, the kth term from the left is 1 if and only if the kth term from the right is not 1. (Hint: *Pascal's triangle.*)

Solution: Upon examining the sequences carefully, one might conjecture that each S_n can be broken into n substrings, such that they exhibit the property of Pascal's triangle, but with concatenation instead of addition.

$$\begin{array}{c} 1 \\ 1, \quad 2 \\ 1, \quad 1,2, \quad 3 \\ 1, \quad 1,1,2, \quad 1,2,3, \quad 4 \\ 1, \quad 1,1,1,2, \quad 1,1,2,1,2,3, \quad 1,2,3,4, \quad 5 \\ \vdots \end{array}$$

In order to prove this, we define the following function on strings:

$$f(x_1, x_2, \ldots, x_m) := 1, 2, \ldots, x_1, 1, 2, \ldots, x_2, \ldots, 1, 2, \ldots, x_m.$$

In particular, the string S_n is defined by $S_n = f(S_{n-1}) \oplus n$ where $\oplus$ represents concatenation. Now for each n, inductively define the sequence of substrings A_{nk} for $k = 1, \ldots, n$ by

$$A_{nk} = f(A_{(n-1)k}), \quad k = 1, \ldots, n-1, \qquad A_{nn} = n.$$

We will prove the following statements by induction on n:

(a) $S_n = A_{n1} \oplus A_{n2} \oplus \cdots \oplus A_{nn}$;

(b) $A_{nk} = A_{(n-1)(k-1)} \oplus A_{(n-1)k}$ for $k = 2, \ldots, n-1$;

(c) For $k = 1, \ldots, n$, A_{nk} has the same length as $A_{n(n+1-k)}$. Furthermore the ith term of A_{nk} from the left is 1 if and only if the ith term of $A_{n(n+1-k)}$ from the right is not 1.

The base cases of $n = 1, 2$ and 3 can be easily verified. Assume the statements hold for S_{n-1} and consider S_n. The first statement (a) follows immediately from the definitions of S_n and A_{nk}.

Now we prove (b), or the 'Pascal's triangle' property conjectured earlier. If $k < n-1$, then

$$\begin{aligned} A_{nk} &= f(A_{(n-1)k}) = f(A_{(n-2)(k-1)} \oplus A_{(n-2)k}) \\ &= f(A_{(n-2)(k-1)}) \oplus f(A_{(n-2)k}) = A_{(n-1)(k-1)} \oplus A_{(n-1)k}. \end{aligned}$$

Otherwise $k = n-1$ and we have

$$\begin{aligned} A_{n(n-1)} &= f(A_{(n-1)(n-1)}) = (1, 2, \ldots, n-1) = (1, 2, \ldots, n-2) \oplus (n-1) \\ &= f(A_{(n-2)(n-2)}) \oplus A_{(n-1)(n-1)} = A_{(n-1)(n-2)} \oplus A_{(n-1)(n-1)}. \end{aligned}$$

In both cases (b) is proven and S_n indeed has the 'Pascal's triangle' property.

It remains to check the symmetry property of the last statement (c). If $k = 1$ or n, then (c) is trivial since $A_{n1} = 1$ and $A_{nn} = n$. If $1 < k < n$, then use (b) to write

$$A_{nk} = A_{(n-1)(k-1)} \oplus A_{(n-1)k}, \quad A_{n(n+1-k)} = A_{(n-1)(n-k)} \oplus A_{(n-1)(n+1-k)}.$$

Then (c) can be reduced to the induction hypothesis on S_{n-1}. This completes the induction.

Finally, to finish the problem, it suffices to note that the required symmetry property of S_n follows immediately from (a) and (c).

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.