Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 35. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, University of Ballarat, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 35 is 1 February 2014. The solutions to Puzzle Corner 35 will appear in Puzzle Corner 37 in the May 2014 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Row of reciprocals

Harry writes down a strictly increasing sequence of one hundred positive integers. He then writes down the reciprocals of the integers.

(i) Is it possible for the sequence of reciprocals to form an arithmetic progression?

(ii) Apart from the last two reciprocals, is it possible for each reciprocal to be the sum of the next two?

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This puzzle corner is also featured on the Mathematics of Planet Earth Australia website http://mathsofplanetearth.org.au/

(iii) Would the answers to the previous questions change if Harry had started with an infinite sequence instead?

Pebble placement

(i) There are several pebbles placed on an $n \times n$ chessboard, such that each pebble is inside a square and no two pebbles share the same square. Perry decides to play the following game. At each turn, he moves one of the pebbles to an empty neighbouring square. After a while, Perry notices that every pebble has passed through every square of the chessboard exactly once and has come back to its original position.

Prove that there was a moment when no pebble was on its original position.

(ii) Peggy aims to place pebbles on an $n \times n$ chessboard in the following way. She must place each pebble at the centre of a square and no two pebbles can be in the same square. To keep it interesting, Peggy makes sure that no four pebbles form a non-degenerate parallelogram.

What is the maximum number of pebbles Peggy can place on the chessboard?

Flawless harmony

Call a nine-digit number *flawless* if it has all the digits from 1 to 9 in some order. An unordered pair of flawless numbers is called *harmonious* if they sum to 987654321. Note that (a, b) and (b, a) are considered to be the same unordered pair.

Without resorting to an exhaustive search, prove that the number of harmonious pairs is odd.

Balancing act

There are some weights on the two sides of a balance scale. The mass of each weight is an integer number of grams, but no two weights on the same side of the scale share the same mass. At the moment, the scale is perfectly balanced, with each side weighing a total of W grams. Suppose W is less than the number of weights on the left multiplied by the number of weights on the right.

Is it always true that we can remove some, but not all, of the weights from each side and still keep the two sides balanced?

Solutions to Puzzle Corner 33

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 33 is awarded to Joe Kupka. Congratulations!

Same sum

Let S be a set of 10 distinct positive integers no more than 100. Prove that S contains two disjoint non-empty subsets which have the same sum.

Solution by Shaun De Roza: First we note that S has $2^{10} - 2 = 1022$ different subsets, excluding the empty set and S itself. Since each number is no more than 100, any subset of S has a sum of no more than 1000. So we have 1022 possible subsets with less than 1000 possible sums. Thus by the pigeonhole principle, two different subsets must have the same sum.

Finally, if the two sum-sharing subsets have common elements, we can simply remove these to form two distinct subsets with the same sum, as required.

Knights and knaves

In the following problems, knights *always tell the truth and* knaves *always lie.*

(i) *There is a queue of people, each of whom is either a knight or a knave. It is known that there are more knights than knaves. Apart from the first person, every person points to someone in front of them in the queue and declares the status of that person (being a knight or a knave). Is it possible for a bystander to determine the actual status of everyone in the queue?*

(ii) *There is a group of people, each of whom is either a knight or a knave. Each person makes the following two statements: 'All my acquaintances know each other', and 'Among my acquaintances, the number of knights is no more than the number of knaves.' We assume that knowing is mutual. Prove that the number of knaves in the group is no more than the number of knights.*

Solution by Joe Kupka: (i) Suppose a person X declares the status of the person Y. If the declaration is 'knight', then X and Y must have the same status. If the declaration is 'knave', then X and Y must have opposite status. Either way, the status of X is determined by the status of Y.

Now for any person X_1, there exist a unique sequence of people

$$X_1 \to X_2 \to X_3 \to \cdots \to F$$

such that each person declares the status of the next. Here F represents the person at the front of the queue, who is the only one not making any declarations. From the earlier arguments, we see that the status of X_1 is determined by the status of X_2, which is determined by the status of X_3 and so on. Hence the status of X_1 is uniquely determined by the status of F. Since the choice of X_1 was arbitrary, we see that everyone's status is determined by F.

So there are two opposite possibilities, depending on the status of F. We must either have a knights and b knaves, or b knights and a knaves, for some a and b which depend on the declarations. But since there are more knights than knaves, the larger of a and b must correspond to the number of knights, eliminating one of the possibilities. Therefore everyone's status can be uniquely determined.

(ii) From the first statement, we can conclude that:

(A) If a knight knows two people, then those two people also know each other.

So amongst the knights, 'knowing' is an equivalence relation. Thus we can partition the knights into knight groups $C_1, C_2, \ldots, C_k$, so that each knight knows everyone in his own knight group, but no-one from the other knight groups.

For each $i = 1, \ldots, k$, define the knave group D_i to be the set of knaves who knows at least one knight from C_i. Again by (A), everyone in $C_i \cup D_i$ must know each other. From the second statement, we can conclude that:

(B) Each knave knows strictly more knights than knaves.

In particular, each knave must know at least one knight and thus belongs to some knave group D_i. Although a knave may belong to multiple knave groups.

Now let m be the largest integer between 0 and k such that

$$\left|\bigcup_{i=1}^{m} C_i\right| \geq \left|\bigcup_{i=1}^{m} D_i\right|. \tag{1}$$

Note that if $m = 0$, then $\bigcup_{i=1}^{m} C_i$ and $\bigcup_{i=1}^{m} D_i$ are empty sets. If $\bigcup_{i=1}^{m} D_i$ contains all the knaves, then (1) implies that the number of knights is greater than or equal to the number of knaves and we are done.

For the sake of contradiction, suppose there exists a knave V who does not belong to $\bigcup_{i=1}^{m} D_i$. We may relabel the knave groups so that V belongs to say, the knave groups $D_{m+1}, \ldots, D_n$. In particular, this means that the set of knights V knows is given exactly by $\bigcup_{i=m+1}^{n} C_i$, while the set of knaves V knows contains $\bigcup_{i=m+1}^{n} D_i \setminus \{V\}$. Applying (B), we must have

$$\left|\bigcup_{i=m+1}^{n} C_i\right| \geq \left|\bigcup_{i=m+1}^{n} D_i\right|. \tag{2}$$

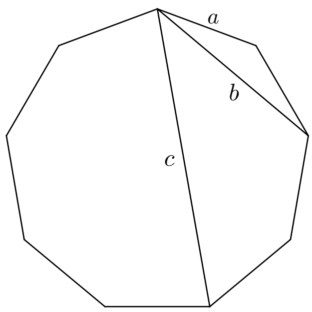
Finally, combining (1) and (2), as well as using the fact that the knights groups C_i are pairwise disjoint, we have

$$\left|\bigcup_{i=1}^{n} C_i\right| = \left|\bigcup_{i=1}^{m} C_i\right| + \left|\bigcup_{i=m+1}^{n} C_i\right| \geq \left|\bigcup_{i=1}^{m} D_i\right| + \left|\bigcup_{i=m+1}^{n} D_i\right| \geq \left|\bigcup_{i=1}^{n} D_i\right|.$$

Since $n > m$, this contradicts the maximality of m. Therefore $\bigcup_{i=1}^{m} D_i$ must contain all the knaves in the first place, completing the proof.

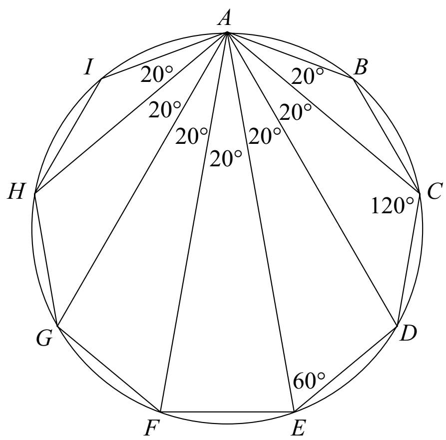
Diagonal difference

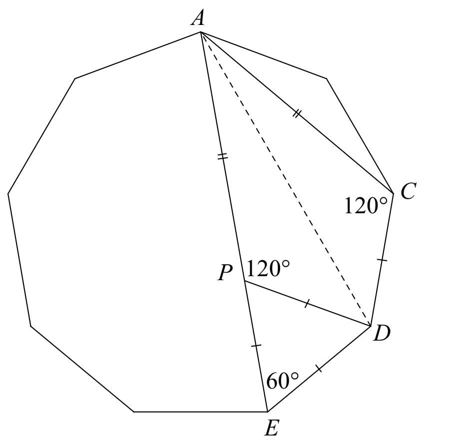
In a regular nonagon, prove that the length difference between the longest diagonal and the shortest diagonal is equal to the side length. In other words, prove $c-b=a$ *in the diagram below.*



Solution by Dave Johnson: Label the vertices of the regular nonagon clockwise from A to I as shown in the diagrams to follow. It is well known that each angle of a regular nonagon is 140°. By inscribing it in a circle, we see that the sides BC, CD, DE, EF, FG, GH and HI all subtend equal angles at A. So these angles must all be $140°/7 = 20°$. By looking at the isosceles triangles ABC and AEF, we can easily deduce that

$$\angle ACD = 140° - 20° = 120° \quad \text{and} \quad \angle AED = 140° - \frac{180° - 20°}{2} = 60°.$$





Now reflect the triangle ACD about the line AD to obtain the triangle APD. Since $\angle PAD = \angle CAD = \angle EAD = 20°$, the point P must lie on AE. Furthermore, since $\angle EPD = 180° - 120° = 60° = \angle DEP$, triangle CPD must be equilateral. Therefore, the required diagonal difference is given by

$$AE - AC = AE - AP = PE = DE,$$

which is indeed the side length of the nonagon.

Scissors and shapes 2

Edward is playing with scissors again. At each move, he chooses a polygon in front of him, and cuts it into two polygons with a single straight cut. Starting with a single rectangle, determine the minimal number of cuts required to obtain, among other shapes, at least 106 polygons with exactly 22 sides.

Solution: For a polygon with s sides, let us define its *score* by $s - 3$. In particular, any triangle has a score of zero while other polygons have positive scores. If there are P polygons with a total of S sides, the *total score* is given by $S - 3P$.

Now consider how a straight cut can affect the total score.

- If the cut joins two vertices, then S increases by 2, P increases by 1 and the score decreases by 1.
- If the cut joins one vertex and one edge, then S increases by 3, P increases by 1 and the score stays the same.
- If the cut joins two edges, then S increases by 4, P increases by 1 and the score increases by 1.

The initial state of a single rectangle has a total score of 1. The final state, which contains 106 polygons with 22 sides as well as other polygons, must have a score of at least

$$106 \times (22 - 3) = 2014.$$

Since the total score increases by at most 1 each cut, we need at least 2013 cuts.

Finally, we show that 2013 cuts are sufficient. The first 22-gon can be made from the starting rectangle using 18 cuts. The other 22-gons can be made from triangles, each requiring 19 cuts. In total, $18 + 105 \times 19 = 2013$ cuts are used. Therefore the answer is indeed 2013.

Perfect recovery

There are n distinct non-negative integers written on the board. Jack memorises these numbers before erasing them and replacing them with the $\binom{n}{2}$ pairwise sums. Now Jill enters the room and studies the sums on the board. Find all positive integers n for which it is possible for Jill to recover the original n integers uniquely.

Solution: Denote the initial set of n distinct non-negative integers by $A = \{a_1, a_2, \ldots, a_n\}$. Suppose that there exists a different set of n distinct non-negative integers $B = \{b_1, b_2, \ldots, b_n\}$ such that the $\binom{n}{2}$ pairwise sums of B are identical to the pairwise sums of A. We shall prove that this is only possible if n is a power of 2.

Define the polynomials

$$P(x) = x^{a_1} + x^{a_2} + \cdots + x^{a_n} \quad \text{and} \quad Q(x) = x^{b_1} + x^{b_2} + \cdots + x^{b_n}.$$

Note the identity

$$P(x)^2 = \sum_{i=1}^{n} x^{2a_i} + 2\sum_{i<j} x^{a_i+a_j} = P(x^2) + 2\sum_{i<j} x^{a_i+a_j}.$$

From the earlier condition on A and B, we have the following equality

$$P(x)^2 - P(x^2) = 2\sum_{i<j} x^{a_i+a_j} = 2\sum_{i<j} x^{b_i+b_j} = Q(x)^2 - Q(x^2).$$

Hence the polynomial $P(x)^2 - Q(x)^2 - P(x^2) + Q(x^2)$ is identically zero.

But since A and B are different sets, the polynomial $P(x) - Q(x)$ is not identically zero. Also since $P(1) - Q(1) = n - n = 0$, we see that $x - 1$ is a factor of $P(x) - Q(x)$. This motivates writing $P(x) - Q(x)$ in the form of

$$P(x) - Q(x) = (x-1)^k R(x),$$

where $k \geq 1$ and $R(1) \neq 0$. Applying this to the polynomial $P(x)^2 - Q(x)^2 - P(x^2) + Q(x^2)$, we have

$$\begin{aligned} &P(x)^2 - Q(x)^2 - P(x^2) + Q(x^2) \\ &= (P(x) + Q(x))(x-1)^k R(x) - (x^2-1)^k R(x^2) \\ &= (x-1)^k((P(x) + Q(x))R(x) - (x+1)^k R(x^2)). \end{aligned}$$

Hence $(P(x) + Q(x))R(x) - (x+1)^k R(x^2)$ is also identically zero. But if we substitute $x = 1$ and use the fact that $R(1) \neq 0$, it implies

$$2nR(1) - 2^k R(1) = 0 \quad \Longrightarrow \quad n = 2^{k-1}.$$

Therefore n must be a power of 2.

It remains to construct examples of A and B with identical pairwise sums in the case of $n = 2^m$. Consider the set of numbers $S = \{0, 1, \ldots, 2^{m+1} - 1\}$ in binary. We may partition S into two sets of size 2^m according to the parity of digit sums. Let the set with even digit sums be A and the set with odd digit sums be B. For example, when $n = 2^3$, we have

$$\begin{aligned} A &= \{0000,\ 0011,\ 0101,\ 0110,\ 1001,\ 1010,\ 1100,\ 1111\}, \\ B &= \{0001,\ 0010,\ 0100,\ 0111,\ 1000,\ 1011,\ 1101,\ 1110\}. \end{aligned}$$

Now for any two elements $x, y \in A$ (with even digit sums), consider the right-most digit which have different values in x and y. Swapping that digit between x and y while preserving the other digits, we arrive at two new numbers w and z which belong to B (with odd digit sums). For example, comparing the elements $x = 0011$ and $y = 1001$ from A, the right-most digit with different values is the third digit. Swapping the third digit, we arrive at $w = 0001$ and $z = 1011$ which are elements of B. Furthermore we must have that $x + y = w + z$.

It is clear the map $(x, y) \to (w, z)$ is a bijection which preserves $x + y = w + z$. Therefore A and B, as constructed, must have identical pairwise sums. It is not possible to uniquely recover either from the pairwise sums.

In conclusion, it is always possible to recover the original set from the pairwise sums if and only if n is not a power of 2.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.