The 54th International Mathematical Olympiad, Santa Marta, Colombia

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The 54th International Mathematical Olympiad (IMO) was held 18–28 July in Santa Marta, Colombia. This is the second country in South America to have hosted an IMO.[1] A total of 527 high school students from 97 countries participated.

Each country sends a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. This Jury was chaired by Maria Falk de Losada.[2]

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 27 problems from 149 problem proposals submitted by 50 of the participating countries from around the world. During the Jury meetings, four of the shorlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. A proposal by UK Leader Geoff Smith, was tried this year. The proposal stipulated that all four major areas of algebra, combinatorics, geometry and number theory be represented among the two easy and two medium problems. The idea being that since the two difficult problems are usually quite inaccessible to most contestants, then at least the four more accessible problems would provide a balanced contest. In hindsight this worked quite well, and may well be tried again next year. Eventually, the Jury finalised the exam questions and then made translations into all the more than 50 languages required by the contestants.

The six questions are described as follows.

1. An easy but novel number theory problem proposed by Japan.
2. A medium combinatorial geometry problem proposed by Australia. This problem requires no technical background whatsoever, only a couple of simple original ideas.

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This article will also be published in Mathematical Contests 2013, The Australian Scene (to appear in 2014) and World Federation of National Mathematics Contests (about to appear).

[1]Argentina hosted the IMO in 1997 and in 2012.

[2]It is noteworthy that three generations of a single family were involved in the successful running of this year's IMO. As mentioned Maria Falk de Losada, amongst other things, chaired the Jury meetings. Her daughter, Maria Elizabeth Losada, played key roles on the Organising Committee. And Maria Elizabeth's 12-year-old daughter, Isabella Mijares, helped out by being a microphone runner for some of the Jury meetings.

3. A difficult classical geometry problem proposed by Russia.
4. A very easy classical geometry problem proposed by South Africa.
5. A medium functional inequality proposed by Bulgaria. One is required to investigate a function that is simultaneously super-additive and sub-multiplicative.
6. This very difficult but most beautiful combinatorial number theory problem was proposed by Russia. It asked one to count the number of permutations of the remainders modulo n in a circle which satisfy a certain arithmetical property. The various solutions to this problem basically amount to a short professional mathematical paper.

These six questions were posed in two exam papers held on Tuesday 23 July and Wednesday 24 July. Each paper had three problems. The contestants worked individually. They were allowed $4\frac{1}{2}$ hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

For many years now there has been an Opening Ceremony prior to the first day of competition. Following the formal speeches there was the parade of the Teams, flanked by Colombian dancers. At the conclusion of the Opening Ceremony the 2013 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes that had been agreed to earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brings something to their attention in a contestant's exam script which is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores.

Question 4 turned out to be very easy as expected. It averaged 5.4 points. Being hard to train for, question 2 mixed things up somewhat. No country achieved a team perfect score for this question. Yet for some students, this was the only question they could solve. As expected, question 6 was very difficult, averaging just 0.3 points. Only seven students scored full marks on this question.

There were 278 (=52.8%) medals awarded, a little more generous than usual. The distributions[3] being 141 (=26.8%) Bronze, 92 (=17.5%) Silver and 45 (=8.5%) Gold. No student achieved the perfect score of 42. However, two students, Yutao Liu of China and Eunsoo Jee of South Korea jointly topped the IMO with outstanding scores of 41 points each. The medal cuts were set at 31 for Gold, 24 for Silver and 15 for Bronze. These awards were presented at the Closing Ceremony. Of

[3]The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of gold, silver and bronze medals must be approximately in the ratio 1:2:3.

those who did not get a medal, a further 141 contestants received an Honourable Mention for solving at least one question perfectly.[4]

Congratulations to the Australian IMO team on their outstanding performance this year. They finished equal 15th in the unofficial country rankings with a clean sweep of medals. Their solid performance gained one gold medal, two silver medals and three bronze medals. Of particular note is the performance of Alex Gunning, Year 10, Glen Waverley Secondary College, Victoria. He received full marks on five of the six problems, and finished 8th in the individual rankings. To put this achievement in perspective, only five other Australians have ever finished in the top 10 at the IMO.[5] Furthermore, he was one of only seven contestants who achieved a perfect score on the most difficult problem 6. With two members of this year's Team eligible for selection for the 2014 IMO Team, things are looking good.

Congratulations also to Australia's Deputy Leader, Ivan Guo. He was the proposer of what was eventually selected as problem 2 on the IMO papers. It is a rare honour to have one of your problem proposals make it onto the IMO. Australia has had a very good run in recent years, now having had three problems on the IMO in the last five years.

The 2013 IMO was organised by the Colombian Mathematics Olympiad, along with the generous support of the University Antonio Nariño.

Venues for future IMOs have been secured up to 2018 as follows: 2015, Thailand; 2016, Hong Kong; 2017, Brazil; 2018, Romania.

The 2014 IMO is scheduled to be held July 3-13 in Cape Town, South Africa.

Much of the statistical information found in this report can also be found at the official website of the IMO (www.imo-official.org).

IMO Papers

Day 1, Tuesday 23 July 2013

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\cdots\left(1 + \frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into

[4]Fourteen contestants managed the feat of what might be called a 'double honourable mention'. They did not get a medal, but solved two questions perfectly, with no marks on any other question.
[5]These are: Andrew Hassell (7th in 1985), Ben Burton (8th in 1992), Geoffrey Chu (4th in 1999), Peter McNamara (10th in 2001) and Andrew Elvey Price (10th in 2009).

several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled. *(The* excircle *of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.)*

Language: English *Time: 4 hours and 30 minutes*
Each problem is worth 7 points

Day 2, Wednesday 14 July 2013

Problem 4. Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;
(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;
(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c. Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

Language: English *Time: 4 hours and 30 minutes*
Each problem is worth 7 points

Results

Mark distribution by question

Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	118	229	438	82	235	481
1	96	32	10	16	84	15
2	9	65	15	14	33	6
3	6	33	16	14	11	6
4	14	22	0	2	0	2
5	3	12	3	5	10	6
6	5	16	4	9	19	4
7	276	118	41	385	135	7
Total	527	527	527	527	527	527
Mean	4.1	2.5	0.8	5.4	2.5	0.3

Some country scores

Rank	Country	Score
1	China	208
2	South Korea	204
3	USA	190
4	Russia	187
5	North Korea	184
6	Singapore	182
7	Vietnam	180
8	Taiwan	176
9	UK	171
10	Iran	168
11	Canada	163
11	Japan	163
13	Israel	161
13	Thailand	161
15	Australia	148

Rank	Country	Score
16	Ukraine	146
17	Mexico	139
17	Turkey	139
19	Indonesia	138
20	Italy	137
21	France	136
22	Belarus	134
22	Hungary	134
22	Romania	134
25	Netherlands	133
26	Peru	132
27	Germany	127
28	Brazil	124
29	India	122
30	Croatia	119

Australian scores at the 2013 IMO

Name	Q1	Q2	Q3	Q4	Q5	Q6	Score	Award
Alexander Chua	7	3	0	7	1	0	18	Bronze
Alex Gunning	7	7	2	7	7	7	37	Gold
Jason Kwong	7	7	0	7	7	0	28	Silver
Seyoon Ragavan	7	7	0	7	1	0	22	Bronze
Rachel Wong	7	1	0	7	3	0	18	Bronze
Jonathan Zheng	7	3	1	7	7	0	25	Silver
Totals	42	28	3	42	26	7	148	
Australian Average	7.0	4.7	0.5	7.0	4.3	1.2	24.7	
IMO Average	4.1	2.5	0.8	5.4	2.5	0.3	15.6	

The medal cuts were set at 31 for gold, 24 for silver and 15 for bronze.

Distribution of awards at the 2013 IMO

Country	Total	Gold	Silver	Bronze	H.M.
Argentina	46	0	0	1	1
Armenia	88	0	1	1	4
Australia	148	1	2	3	0
Austria	77	0	1	1	2
Azerbaijan	73	0	0	2	3
Bangladesh	60	0	0	3	1
Belarus	134	1	2	3	0
Belgium	82	0	1	2	2
Bolivia	5	0	0	0	0
Bosnia & Herzegovina	56	0	0	1	4
Brazil	124	0	3	1	2
Bulgaria	101	0	1	2	3
Canada	163	2	2	2	0
Chile	35	0	0	1	2
China	208	5	1	0	0
Colombia	77	0	0	2	2
Costa Rica	59	0	0	1	5
Croatia	119	2	0	2	2
Cuba	11	0	0	0	1
Cyprus	52	0	0	1	3
Czech Republic	108	1	0	3	2
Denmark	31	0	0	0	3
Ecuador	45	0	0	1	2
El Salvador	14	0	0	0	2
Estonia	67	0	0	2	3
Finland	46	0	1	0	0
France	136	0	2	4	0
Georgia	75	0	0	2	4
Germany	127	0	2	4	0
Greece	101	0	2	1	3

Distribution of awards at the 2013 IMO (continued)

Country	Total	Gold	Silver	Bronze	H.M.
Honduras	0	0	0	0	0
Hong Kong	117	0	1	5	0
Hungary	134	0	2	4	0
Iceland	27	0	0	0	2
India	122	0	2	3	0
Indonesia	138	1	1	4	0
Iran	168	2	3	1	0
Ireland	33	0	0	0	4
Israel	161	1	3	2	0
Italy	137	1	2	1	2
Japan	163	0	6	0	0
Kazakhstan	116	0	1	4	1
Kosovo	25	0	0	0	3
Kyrgyzstan	36	0	0	1	2
Latvia	47	0	0	1	3
Liechtenstein	15	0	0	1	0
Lithuania	78	0	0	3	3
Luxembourg	25	0	0	1	0
Macedonia (FYR)	34	0	0	1	1
Malaysia	117	0	2	3	0
Mexico	139	0	3	3	0
Moldova	71	0	0	2	4
Mongolia	84	0	0	3	3
Montenegro	1	0	0	0	0
Morocco	17	0	0	0	1
Netherlands	133	0	2	3	1
New Zealand	77	0	0	2	3
Nicaragua	22	0	0	0	3
Nigeria	18	0	0	1	0
North Korea	184	2	4	0	0
Norway	36	0	0	1	1
Pakistan	25	0	0	0	3
Panama	19	0	0	0	2
Paraguay	38	0	0	2	1
Peru	132	0	3	2	1
Philippines	72	0	0	3	2
Poland	79	0	1	1	3
Portugal	111	1	0	4	1
Puerto Rico	14	0	0	0	1
Romania	134	0	3	3	0
Russia	187	4	2	0	0
Saudi Arabia	84	0	0	4	0
Serbia	112	1	1	2	2
Singapore	182	1	5	0	0
Slovakia	112	0	1	3	2
Slovenia	34	0	0	0	4

Distribution of awards at the 2013 IMO (continued)

Country	Total	Gold	Silver	Bronze	H.M.
South Africa	64	0	0	2	3
South Korea	204	5	1	0	0
Spain	63	0	0	2	3
Sri Lanka	65	0	0	1	4
Sweden	62	0	1	1	2
Switzerland	88	0	0	3	2
Syria	14	0	0	0	1
Taiwan	176	2	4	0	0
Tajikistan	65	0	0	1	4
Thailand	161	1	4	1	0
Trinidad & Tobago	16	0	0	0	1
Tunisia	49	0	0	1	3
Turkey	139	1	2	3	0
Turkmenistan	78	0	0	4	1
Uganda	1	0	0	0	0
Ukraine	146	1	3	1	1
United Kingdom	171	2	3	1	0
United States of America	190	4	2	0	0
Uruguay	7	0	0	0	0
Venezuela	9	0	0	0	1
Vietnam	180	3	3	0	0
Total (97 teams, 527 contestants)		**45**	**92**	**141**	**141**

NB: Not all countries sent a full team of six students.