Lift-Off Fellowship report: A strong Oka principle for circular domains

Tyson Ritter*

In complex geometry, *Stein manifolds* are of fundamental importance as those complex manifolds with a rich supply of holomorphic (that is, complex differentiable) functions from them to the complex numbers $\mathbb{C}$. When studying holomorphically defined problems on Stein manifolds, an interesting phenomenon can arise in which the existence of a continuous solution is enough to give a holomorphic solution. This is somewhat surprising as holomorphic maps are much more rigid than continuous maps, so we would not necessarily expect that the only obstruction to solving a holomorphic problem is topological in nature. In such instances we say that the *Oka principle* holds, named for Kiyoshi Oka who gave one of the first results of this kind in 1939, showing that the holomorphic and topological classifications of line bundles over a Stein manifold are the same.

While it has long been clear that the suitable notion of a manifold having many holomorphic functions into $\mathbb{C}$ is that of being Stein, it has only recently become clear what an appropriate dual notion should be. Following Gromov's seminal paper on the Oka principle in 1989 [3], Forstnerič [1] recently introduced the class of *Oka manifolds* as those satisfying a number of equivalent so-called *Oka properties*, each stating in some way that there are many holomorphic maps from $\mathbb{C}$ into the manifold (see [2] for a recent survey). One of the simplest such statements is Gromov's Oka property, that every continuous map from a Stein manifold into an Oka manifold can be continuously deformed into a holomorphic map.

There are still many open questions relating to Oka manifolds. For instance, the embedding theorem for Stein manifolds states that every Stein manifold S of dimension n can be embedded (properly holomorphically) as a closed submanifold of $\mathbb{C}^N$, for $N = 2n + 1$. In this case the embedding is clearly a homotopy equivalence between S and its image, yet we can relax this and ask that the embedding instead be a homotopy equivalence between S and the entire target manifold. We call such an embedding *acyclic*. Of course, as soon as S has non-trivial topology we need to allow more general targets than $\mathbb{C}^N$, the most natural generalisation being to consider targets that are, like $\mathbb{C}^N$, both Stein and Oka. The question then becomes, does every Stein manifold have an acyclic embedding into a Stein Oka manifold? Additional motivation for this question arises from the holomorphic homotopy theory of Lárusson [4].

* School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia.
Email: tyson.ritter@adelaide.edu.au

During my PhD I considered this question in the simplest non-trivial case, namely for 1-dimensional Stein manifolds, which are precisely the open (that is, non-compact) Riemann surfaces. I proved that every open Riemann surface with abelian fundamental group acyclically embeds into a 2-dimensional Stein Oka manifold [5], and more generally that every open Riemann surface acyclically embeds into an Oka manifold [6]. In the more general setting the question of whether the targets are Stein remains unanswered, and appears to be very difficult.

An important ingredient in these results is the following *strong Oka principle*, named such as it is a strengthening of Gromov's Oka principle for the special case of maps from circular domains into the Stein Oka manifold $\mathbb{C} \times \mathbb{C}^*$ (where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). A *circular domain* is any domain in $\mathbb{C}$ given as the open unit disc from which a finite number of smaller pairwise disjoint closed discs of positive radii have been removed.

Strong Oka Principle. ([5].) *Every continuous map from a circular domain to* $\mathbb{C} \times \mathbb{C}^*$ *can be continuously deformed into a proper holomorphic embedding.*

During my Lift-Off Fellowship I focused on extending the strong Oka principle to the case of punctured circular domains, that is, circular domains from which a finite number of points have been removed. The simplest such case is the unit disc with a finite number of punctures and no holes. While it is easy to handle a single puncture, the problem becomes considerably more difficult as soon as two punctures are permitted, with difficulties arising when the punctures are both extremely close to the origin. In this case it is possible to show that some homotopy classes of continuous maps contain proper embeddings, but difficulties arise and further investigation is needed when the winding numbers about the two punctures are large and opposite in sign.

I will continue my research on punctured circular domains, together with other related problems in Oka theory, during my time as an ARC Research Associate at the University of Adelaide with Associate Professor Finnur Lárusson. I wish to thank the AustMS for the Lift-Off Fellowship that has enabled me to lay important groundwork for my future research, and to the University of Adelaide for support during my time as a Lift-Off Fellow. I strongly encourage all PhD students to consider applying for the Lift-Off Fellowship after submitting their thesis.

References

[1] Forstnerič, F. (2009). Oka manifolds. *C. R. Math. Acad. Sci. Paris* **347,** 1017–1020.

[2] Forstnerič, F. and Lárusson, F. (2011). Survey of Oka theory. *New York J. Math.* **17a,** 11–38.

[3] Gromov, M. (1989) Oka's principle for holomorphic sections of elliptic bundles. *J. Amer. Math. Soc.* **2,** 851–897.

[4] Lárusson, F. (2004). Model structures and the Oka principle. *J. Pure Appl. Algebra* **192,** 203–223.

[5] Ritter, T. (2013). A strong Oka principle for embeddings of some planar domains into $\mathbb{C}\times\mathbb{C}^*$. *J. Geom. Anal.* **23,** 571–597.
[6] Ritter, T. (2013). Acyclic embeddings of open Riemann surfaces into new examples of elliptic manifolds. *Proc. Amer. Math. Soc.* **141,** 597–603.

Tyson Ritter completed his undergraduate study in 2004 at the University of Adelaide with bachelor degrees in Physics, Electrical Engineering and Pure Mathematics (Honours). After working as a Research Engineer for several years, he returned to study in 2008, obtaining his PhD in Pure Mathematics from the University of Adelaide in 2011 under the supervision of Associate Professor Finnur Lárusson. He now works in the School of Mathematical Sciences at the University of Adelaide as an ARC Research Associate in the areas of complex geometry and complex analysis.