Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 31. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

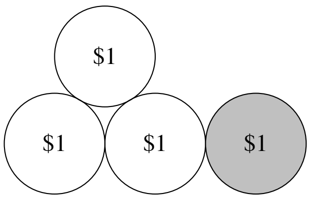
For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, University of Ballarat, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 31 is 1 May 2013. The solutions to Puzzle Corner 31 will appear in Puzzle Corner 33 in the July 2013 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Rolling in riches

Place four $1 coins as shown in the diagram below. Now roll the shaded coin anticlockwise around the other three, touching them the entire time, until it returns to the original position. How much has the shaded coin rotated relative to its centre?



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This puzzle corner is also featured on the Mathematics of Planet Earth Australia website http://mathsofplanetearth.org.au/

Picky padlocks

An ancient scroll is kept in a chest, which is locked by a number of padlocks. All padlocks must be unlocked in order to open the chest. Copies of the keys to the padlocks are distributed amongst 12 knights, such that any group of 7 or more knights can open the chest should they choose to do so, but any group of less than 7 cannot. What is the minimal number of padlocks required to achieve this?

Stargazing

An astronomer observed 20 stars with his telescope. When he added up all the pairwise distances between the stars, the result was X. Suddenly a cloud obscured 10 of the stars. Prove that the sum of the pairwise distances between the 10 remaining stars is less than $\frac{1}{2}X$.

Bonus: Can you improve the bound? What is the smallest real number r such that the new sum is always less than rX, regardless of the configuration of the stars?

Golden creatures

Submitted by Joe Kupka

At the beginning of time, in a galaxy far, far away, the Queen of Heaven gives birth to 40 golden creatures. On the last day of each year the King of Heaven sacrifices a randomly chosen creature to his own glory. After every 20 sacrifices, the Queen gives birth to 20 new creatures. Every creature lives until it is sacrificed by the King. Any creature who reaches 100 years of age receives a congratulatory letter from the Queen.

(i) What is the probability that a creature will receive a congratulatory letter?

(ii) How many congratulatory letters, on average, will the Queen write in the first 1000 years?

(iii) One of the first 40 creatures is named Adam. One of the 20 creatures born after 40 sacrifices is named Eve. What is the probability that Adam will outlive Eve?

Uncovered construction

Can you construct a set of 100 rectangles, with the property that not one of the rectangles can be completely covered by the the other 99?

Solutions to Puzzle Corner 29

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 29 is awarded to Jensen Lai. Congratulations!

Final product

Four real numbers can form six pairwise products. If five of the six products are 2, 3, 4, 5 and 6, what is the sixth product?

Bonus: Given that the four numbers are all positive, what are they?

Solution by Dave Johnson: Let the four numbers be a, b, c and d. Since

$$(ab)(cd) = (ac)(bd) = (ad)(bc),$$

the six pairwise products 2, 3, 4, 5, 6 and x can be grouped into three pairs so that the product of each pair is equal to $abcd$. Now two of these pairs come from four of the five given numbers. The only way to create two equal products is by having $2 \times 6 = 3 \times 4 = 12$. This leaves $5x = 12$, hence the sixth product is 12/5.

Bonus: Without the loss of generality, assume $0 < a \leq b \leq c \leq d$. Then by comparing the size of the products, we have

$$ab = 2, \qquad ac = 12/5, \qquad bd = 5, \qquad cd = 6 \qquad \text{and} \qquad \{ad, bc\} = \{3, 4\}.$$

Thus there are two possibilities:

- If $ad = 3$ and $bc = 4$, then
$$(a, b, c, d) = \left(\sqrt{\tfrac{6}{5}}, 2\sqrt{\tfrac{5}{6}}, 2\sqrt{\tfrac{6}{5}}, 3\sqrt{\tfrac{5}{6}}\right);$$
- If $ad = 4$ and $bc = 3$, then
$$(a, b, c, d) = \left(2\sqrt{\tfrac{2}{5}}, \sqrt{\tfrac{5}{2}}, 3\sqrt{\tfrac{2}{5}}, 2\sqrt{\tfrac{5}{2}}\right).$$

Sticky stalemate

Adrian, Benny, and Christie are running for president in their club. On the ballot, each voter lists the three candidates in order of their preference. Counting only the first preferences results, dramatically, in a three-way tie. To break the deadlock, the second preferences are counted, but again there is a three-way tie.

The astute Adrian notes that, since the number of voters is odd, they can make two-way decisions without ties. He proposes that the voters first choose between Benny and Christie, then the winner faces Adrian for the position.

The brainy Benny thinks it's a good resolution, since they only want to identify the winner, not the runner-up. The clever Christie disagrees and complains that this is giving Adrian an advantage. Who is right? Assuming the voters never change their preferences, what is Adrian's chance of winning under his proposed voting system?

Solution by Joe Kupka: From the conditions of the problem, the number of voters is $3n$ for some odd positive integer n. Each of the candidates A, B and C received n first preferences and n second preferences. Let the number of voters who voted $A > B > C$ be x and the number of voters who voted $A > C > B$ be y. Then $x + y = n$ and the size of the remaining voting groups can be easily computed:

$$\begin{aligned} &A > B > C\colon x, \qquad && A > C > B\colon y,\\ &B > C > A\colon x, && B > A > C\colon y,\\ &C > A > B\colon x, && C > B > A\colon y. \end{aligned}$$

When only comparing B against C, we have

$$B > C\colon x + x + y = n + x, \qquad C > B\colon y + x + y = n + y.$$

Similarly (noting the cyclic symmetry), we have

$$A > B\colon n + x, \qquad B > A\colon n + y; A > C\colon n + y, \qquad C > A\colon n + x.$$

Since n is odd, $x \neq y$. If $x > y$, then B wins against C but goes on to lose against A. If $y > x$, then C beats B then also goes on to lose against A. Therefore Christie is right, Adrian will always win under the proposed system.

Four-way intersection

Start with a unit square. Join each vertex to the midpoints of the two opposite sides. An octagon is formed in the centre. What is the area of the octagon?

Solution by Jensen Lai: Refer to Figures 1 and 2. Since F and H are midpoints, $ABFH$ is a rectangle with area $\frac{1}{2}$. Then

$$\text{Area}(\triangle ABF) = \text{Area}(\triangle ABH) = \tfrac{1}{4}.$$

The diagonals of $ABFH$ intersect at X. Hence X is the midpoint of AF. So the area of $\triangle BFX$ (see Figure 1) is given by

$$\text{Area}(\triangle BFX) = \tfrac{1}{2}\text{Area}(\triangle BFA) = \tfrac{1}{8}.$$

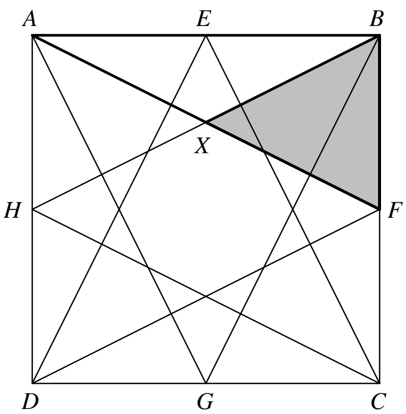


Figure 1

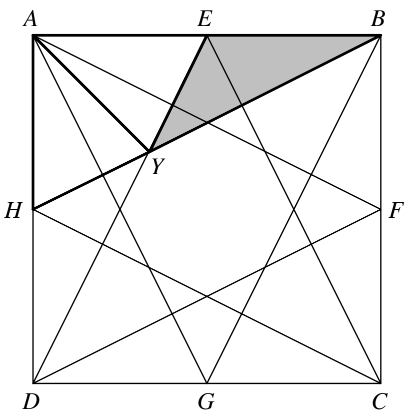


Figure 2

Since E is the midpoint of AB, we have

$$\text{Area}(\triangle BYE) = \text{Area}(\triangle AYE) = \text{Area}(\triangle AYH),$$

where the second equality follows because $\triangle AYE$ and $\triangle AYH$ are symmetric about the line AY. Hence the area of $\triangle BYE$ (see Figure 2) is

$$\text{Area}(\triangle BYE) = \tfrac{1}{3}\text{Area}(\triangle BFH) = \tfrac{1}{12}.$$

Now combining $\triangle BFX$ and $\triangle BYE$, the shaded area in Figure 3 is

$$\text{Area}(\triangle BFX) + \text{Area}(\triangle BYE) = \tfrac{1}{8} + \tfrac{1}{12} = \tfrac{5}{24}.$$

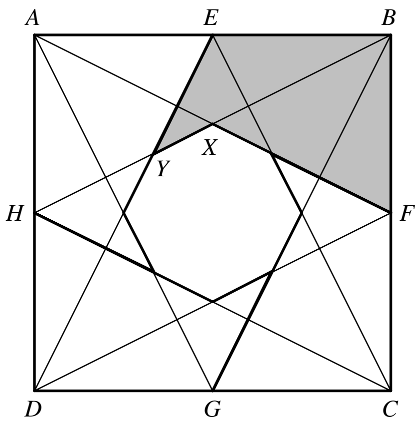


Figure 3

Since the area outside of the octagon can be divided into four copies of the shaded area, as shown in Figure 3, the area of the octagon is $1 - 4 \times \frac{5}{24} = \frac{1}{6}$.

Three-point line

Every point on the real number line is coloured in one of two colours: red or blue.

(i) *Prove that there exist three real numbers $x < y < z$ such that they have the same colour and satisfy $zy = yx$.*

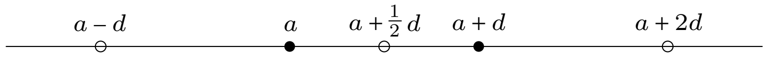
(ii) *Prove that for any positive real number r, there exist three real numbers $x < y < z$ such that they have the same colour and satisfy $zy = r(yx)$.*

Solution by Dave Johnson:

(i) Assume that the assertion is false. Take any two points of the same colour, say red. Let them be at positions a and $a + d$. Consider the triples,

$$(a - d, a, a + d), \qquad (a, a + d, a + 2d) \qquad \text{and} \qquad \left(a, a + \tfrac{1}{2}d, a + d\right).$$

In order to avoid a red triple, the points at $a - d$, $a + 2d$ and $a + \frac{1}{2}d$ must all be blue. But since $a + \frac{1}{2}d$ is the mean of the other two, we have a blue triple, contradicting the assumption.



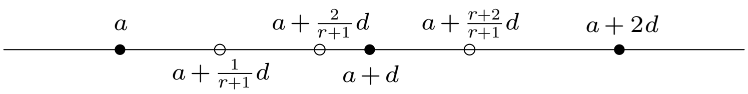
(ii) Again assume that the assertion is false. By part (i), there exist three points $a, a+d$ and $a+2d$ of the same colour, say red. In order to avoid a red triple, the following three weighted means must all be blue:

$$\begin{aligned}\frac{r}{r+1}a + \frac{1}{r+1}(a+d) &= a + \frac{1}{r+1}d,\\ \frac{r}{r+1}(a+d) + \frac{1}{r+1}(a+2d) &= a + \frac{r+2}{r+1}d,\\ \frac{r}{r+1}a + \frac{1}{r+1}(a+2d) &= a + \frac{2}{r+1}d.\end{aligned}$$

But since

$$a + \frac{2}{r+1}d = \frac{r}{r+1}\left(a + \frac{1}{r+1}d\right) + \frac{1}{r+1}\left(a + \frac{r+2}{r+1}d\right),$$

a blue triple is formed. This is a contradiction.



Revolving vault

A vault's door has a circular lock, which has n indistinguishable buttons on its circumference with equal spacing. Each button is linked to a light on the other side of the door, which is not visible from outside the vault. Each button toggles its linked light between on and off.

In the beginning, some number of lights are off and the door is locked. For each move, you are allowed to press several buttons simultaneously. If all lights are turned on as a result, the vault door will open. Otherwise, the circular lock will rotate to a random position, without changing the on/off status of each individual light. The rotation occurs quickly so it is impossible to track how much the lock has rotated.

Prove that it is always possible to open the door using a finite number of moves if and only if n is a power of 2.

Solution by Jensen Lai: We first show that it is not always possible to open the vault if n is not a power of 2. Now n must contain an odd factor m which is greater than 1. Consider a set of m equally spaced buttons around the lock and their associated lights, call them *special*.

Since m is odd, during each simultaneous button press, there must exist a pair of consecutive special buttons which are either both pressed or both not pressed. (Note that if m is even, we could press every second special button to avoid this situation.) So if two consecutive special lights have different initial states, it is possible for them to always have different states. Therefore it is not always possible to turn on all the lights.

Now we show that it is possible to unlock the vault if $n = 2^k$ for some nonnegative integer k. For $n = 2^0$, simply press the only button to turn on the only light and open the vault. For $n = 2^1$, the algorithm is as follows:

- Begin by noting that at least one light is off, as the vault is locked initially. Assume that both lights are off. The first move is to press both buttons. If the assumption was indeed correct, this will open the door.
- If the door does not open, then the assumption was incorrect. Exactly one light is on, both before and after the first move. The second move is to press only one button. This toggles exactly one light, and hence brings both lights to the same state. If both are on, then the door will open.
- If the door still does not open, then both lights are off. The third move is to press both buttons. This will definitely open the vault.

We will now denote each move by a binary string. In particular, for n buttons, we use a string of n digits containing 0s and 1s. The 1s will indicate the buttons to be pressed. For example, the algorithm for $n = 1$ is $A_1 = (1)$, while the algorithm for $n = 2$ is $A_2 = (11, 01, 11)$.

Now we claim that the vault with $n = 2^k$ buttons can be opened in $2^{2^k} - 1$ moves. Proceed by induction on k. The base cases has already been established. Assume the algorithm for opening the vault with $n = 2^k$ buttons is given by

$$A_{2^k} = (M_1,\ M_2,\ \ldots,\ M_{2^{2^k}-1}).$$

For $n = 2^{k+1}$, begin by pairing each button with the directly opposite button, forming 2^k pairs. A pair is called *matched* if they have the same state, and *unmatched* otherwise.

- If all pairs are matched, then we can simply perform A_{2^k} by treating each of the 2^k pairs as a single button. Explicitly, the algorithm is
$$B_{2^{k+1}} = (M_1M_1,\ M_2M_2,\ \ldots,\ M_{2^{2^k}-1}M_{2^{2^k}-1}),$$
where M_1M_1 is the concatenation of two copies of M_1 and so on. This will open the vault.
- If some pairs are unmatched, then we first try to match all pairs. Since pressing only one button in a pair will toggle the 'matched-ness' of the pair, the problem of matching 2^k pairs is similar to the problem of turning on 2^k lights. This is done by performing A_{2^k} on half of the 2^{k+1} buttons, or
$$C_{2^{k+1}} = (OM_1,\ OM_2,\ \ldots,\ OM_{2^{2^k}-1}),$$
where O is the string of 2^k 0s. This will ensure that all pairs are matched at some stage.
- The only problem with $C_{2^{k+1}}$ is that we have no confirmation of when all pairs are matched. So we do not know when to stop $C_{2^{k+1}}$ and start $B_{2^{k+1}}$. To resolve this issue, note that every move in $B_{2^{k+1}}$ preserves the 'matched-ness' of all pairs. So the key is to perform the entirety of $B_{2^{k+1}}$ in between the moves of $C_{2^{k+1}}$, just in case that all pairs are matched at any stage.

Therefore the complete algorithm required to open the vault for $n = 2^{k+1}$ buttons is given by

$$\begin{aligned} A_{2^{k+1}} = (&M_1M_1,\ M_2M_2,\ \ldots,\ M_{2^{2^k}-1}M_{2^{2^k}-1},\ OM_1,\\ &M_1M_1,\ M_2M_2,\ \ldots,\ M_{2^{2^k}-1}M_{2^{2^k}-1},\ OM_2,\\ &\qquad\vdots\\ &M_1M_1,\ M_2M_2,\ \ldots,\ M_{2^{2^k}-1}M_{2^{2^k}-1},\ OM_{2^{2^k}-1},\\ &M_1M_1,\ M_2M_2,\ \ldots,\ M_{2^{2^k}-1}M_{2^{2^k}-1}). \end{aligned}$$

It can be easily checked that $A_{2^{k+1}}$ has exactly $2^{2^{k+1}} - 1$ moves. This completes the induction.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.