Puzzle Corner

Ivan Guo*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 26. Each Puzzle Corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095.

The deadline for submission of solutions for Puzzle Corner 26 is 1 May 2012. The solutions to Puzzle Corner 26 will appear in Puzzle Corner 28 in the July 2012 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Ratio of radii

A sphere with radius r is inscribed in a regular tetrahedron, which is inscribed in a larger sphere with radius R. Find the ratio of R to r.

Counterfeit coins

There are 20 coins in front of you, two of which are counterfeits. The genuine coins are identical in weight, but the counterfeits are slightly lighter. Can you identify 10 genuine coins by using a balance-scale twice? Note that the two counterfeits are not necessarily the same weight as each other.

*School of Mathematics and Statistics, University of Sydney, NSW 2006.
E-mail: ivanguo1986@gmail.com

Poor turnout

Submitted by Dave Johnson

Following the service, the vicar asked the bell-ringer if he could work out the ages of the three people who attended today, given that the product of their ages was 2450 and the sum was twice the age of the bell-ringer. After some thought, the bell-ringer was unable to do so. The vicar then revealed that he (the vicar) was in fact older than all three of them. Upon hearing that, the bell-ringer quickly responded with the three ages. What were the ages of all five people?


Photo: Martyn E. Jones, stock.xchng

Chocolate addiction


Photo: Andre Machado

Willy has several jars filled with chocolates, none of which is empty. Each move, he is allowed to either double the content of one jar, or eat one chocolate from every jar. Can Willy always empty all the jars using these moves?

Bonus: If the doubling move is replaced by tripling, can Willy always empty all the jars?

Odd polygons

Prove that it is never possible to tile a polygon with only odd integer side lengths using 1×2 dominoes.

Probability problems

1. On a circle, n points are chosen randomly. What is the probability that they all lie on a semicircular arc?
2. A strange machine takes a positive integer n as input and randomly outputs an integer between 1 and n inclusive. We start by giving the machine $n = 1000$, and continue to feed the output back into the machine as input. On average, how many times do we have to use the machine until it outputs the number 1?

Solutions to Puzzle Corner 24

Many thanks to everyone who submitted solutions. The \$50 book voucher for the best submission to Puzzle Corner 24 is awarded to Jensen Lai. Congratulations!

Age-old question

'Two days ago, I was 10. But next year I will be 13.' How is this possible?

Photo: Art Brazee

Solution by John Giles, who turned 78 on 31 December 2011: Consider a child who was born on 31 December 2000, making the statement on 1 January 2012. Two days ago, on 30 December 2011, she was 10. But next year, on 31 December 2013, she will be 13.

A tricky alternative by John Harper: Consider a child who got on a plane from Los Angeles to Sydney on the evening of 30 December 2011. She would have flown across the International Date Line and arrived on 1 January 2012. To her perception, 'two days ago' may appear to be 29 December in the United States. So the date of birth could have been either 30 or 31 December 2000.

Dice design

Can you design two different dice so that their sum still behaves like a pair of ordinary dice? That is, there must be one way to get a sum of 2, two ways to get a sum of 3, and so on. A die must have six faces, each labelled with a positive integer.

Solution by Dave Johnson: A pair of dice with faces labelled $\{1, 2, 2, 3, 3, 4\}$ and $\{1, 3, 4, 5, 6, 8\}$ would satisfy the requirements. This can be motivated by considering the frequency polynomial of one normal die:

$$f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 = x(x + 1)(x^2 + x + 1)(x^2 - x + 1).$$

The frequency polynomial of the two dice total is given by

$$f(x)^2 = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$$

It can be alternatively factorised as

$$\begin{aligned} f(x)^2 &= [x(x + 1)(x^2 + x + 1)][x(x + 1)(x^2 + x + 1)(x^2 - x + 1)^2] \\ &= [x + 2x^2 + 2x^3 + x^4][x + x^3 + x^4 + x^5 + x^6 + x^8]. \end{aligned}$$

This is also the only factorisation of $f(x)^2 = g(x)h(x)$ with g, h having no constant terms and $g(1) = h(1) = 6$ (the two properties required for producing valid dice). So there are no other solutions.

Triangular trip

Assume the Earth is perfectly spherical and you are standing somewhere on its surface. You travel exactly 1 kilometre south, then 1 kilometre east, then 1 kilometre north. Surprisingly, you find yourself back at the starting point. If you are not at the North Pole, where can you possibly be?

Solution by John Harper: Since the starting point is not the North Pole, the south and north trips are inverses of each other. So the 1 kilometre east trip must return you to the same spot.

Start 1 kilometre north of the latitudinal circle with circumference $1/n$ kilometres near the South Pole, where n is any positive integer. Travelling 1 kilometre south will reach that circle, 1 kilometre east will go around the circle n times, and 1 kilometre north will return to the starting point.

Author note: Latitudinal circles of circumference $1/n$ kilometres also exist near the North Pole, but they are always less than 1 kilometre south of the North Pole, no matter how big (or small) we assume the Earth is. So the only solutions exist near the South Pole.

Picky toothpicks

Pete and Pat take turns picking up some toothpicks off the floor. Pete goes first, and is allowed to pick up as many as he wants, but not all of them. In each subsequent turn, the person can pick up any number of toothpicks, as long as it doesn't exceed the number picked up in the previous turn. Passing is not allowed. The person who picks up the last toothpick wins. Who wins?

Solution by Peter Nickolas: Let N denote the initial number of toothpicks. If $N = 1$, the rules do not allow the game to be played, so we will assume that $N \geq 2$. We claim that: (a) Pete has a winning strategy if N is a not a power of 2, and (b) Pat has a winning strategy if N is a power of 2.

For (a), any N which is not a power of 2 can be written as $N = 2^s t$ where $s \geq 0$, and $t \geq 3$ is odd. We will inductively prove that Pete has a winning strategy in which the first move is to pick up 2^s toothpicks. This is certainly true for the base case of 3 toothpicks, as Pete wins by picking up 1 toothpick. Now assume Pete has a winning strategy whenever the number of toothpicks is less than N and not a power of 2.

After Pete's first move of picking up 2^s toothpicks, there are $2^s(t-1)$ left. If Pat's first move is to also pick up 2^s toothpicks, then the number remaining is $2^s(t-2)$. If $t-2 = 1$, then Pete has an immediate win by picking up the remaining 2^s toothpicks. Otherwise $t - 2 \geq 3$, so $2^s(t-2) < N$ is not a power of 2. The inductive assumption implies that Pete has a winning strategy from this point by again picking up 2^s toothpicks.

If Pat's first move was instead to pick up $m < 2^s$ toothpicks, then the remaining number of toothpicks cannot be a power of 2 because $2^s(t-1) - m$ is strictly between two positive multiples of 2^s, or

$$2^s(t-2) < 2^s(t-1) - m < 2^s(t-1).$$

Rewrite it as $2^s(t-1) - m = 2^{s'}t'$ where $s' \geq 0$ and $t' \geq 3$ is odd. Then we must have $s' < s$ and $2^{s'}$ divides m. In particular, $2^{s'} \leq m$. Again by the inductive assumption, Pete has a winning strategy here by picking up $2^{s'}$, which is a valid move. Therefore (a) is proven.

For (b), suppose that $N = 2^s$. If Pete starts by picking up at least half of the toothpicks, then Pat has an immediate win by picking up all of the remaining toothpicks. But if Pete picks up less than half of the toothpicks, say m, then $2^s - m$ is clearly not a power of 2. Pat can then use the winning strategy from part (a) by picking up $2^{s'}$ toothpicks where $2^s - m = 2^{s'}t'$. Note that by arguments similar to above, we can check that $2^{s'} \leq m$ so the move is valid.

Invisible soldiers

In the Cartesian plane, the point (x, y) is called a lattice point if x and y are both integers. Suppose that a general is standing at the origin, while there is a soldier standing at every other lattice point. Label a soldier invisible *if the general cannot see him because another soldier is in the way. Prove that it is possible to find arbitrarily large circles in the plane containing only invisible soldiers.*

Solution by Jensen Lai: Select an arbitrarily large integer n. Arrange n^2 distinct prime numbers into an $n \times n$ square. Label each prime $p_{i,j}$ where i and j correspond to the column and row numbers. Let c_i be the product of all primes in column i and r_j be the product of all primes in row j.

Solve the following system of $2n$ equations:

$$X \equiv -i \pmod{c_i},\ 1 \leq i \leq n; \quad Y \equiv -j \pmod{r_j},\ 1 \leq j \leq n.$$

Since $c_1, \ldots, c_n$ are pairwise co-prime and $r_1, \ldots, r_n$ are also pairwise co-prime, by the Chinese Remainder Theorem, we can find X and Y satisfying these equations.

Now, each $X + i$ is a multiple of c_i and each $Y + j$ is a multiple of r_j. So $X + i$ and $Y + j$ share a common factor of $p_{i,j}$, which is a factor of both c_i and r_j. Hence each of the n^2 soldiers standing at the lattice points

$$(X + i, Y + j),\ 1 \leq i, j \leq n$$

will be blocked from the general's view by the soldiers standing at

$$\left(\frac{X+i}{p_{i,j}}, \frac{Y+j}{p_{i,j}}\right),\ 1 \leq i, j \leq n,$$

rendering them invisible.

Therefore, for arbitrarily large n, there exist squares of side length $n-1$ containing only invisible soldiers. By choosing circles inscribed in these squares, there exist arbitrary large circles containing only invisible soldiers.

Curve and Chord

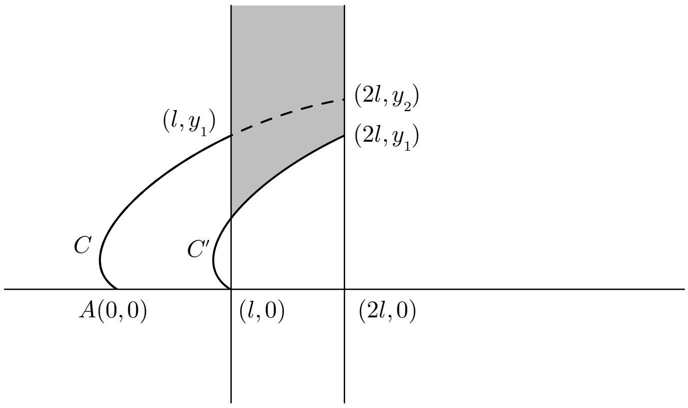
Let A and B be two points on the plane, one unit apart. There is a continuous curve joining A and B. We want to find a chord of the curve, parallel to AB, with length l.

(1) *Show that this is always possible if l is the reciprocal of a positive integer.*

(2) *What if l is not the reciprocal of a positive integer?*

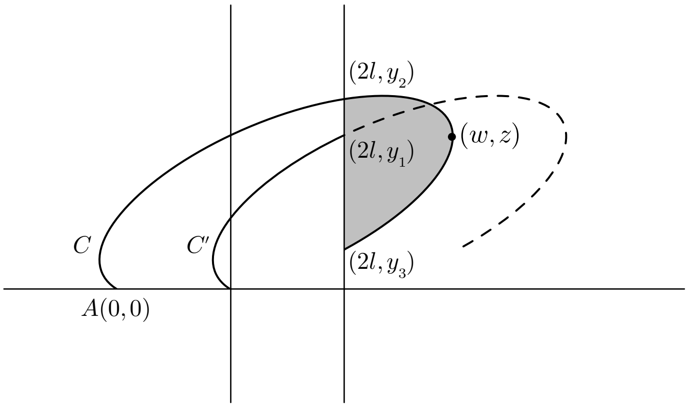
Solution by Jensen Lai:

(1) Let A be $(0,0)$, B be $(1,0)$, and denote the curve by C. Translate C by l units to the right to form the curve C'. If C and C' intersect at the point (x,y), then the chord from $(x-l,y)$ to (x,y) has length l and is parallel to AB. Now assume C and C' do not intersect.
If $l=1$, the curves intersect at B. That's a contradiction.
If $l=1/n$ where $n>1$ is a positive integer, divide the line AB into n equal segments of length l. Follow curve C from point A until its *last* intersection with the line $x=l$. Let the point be (l,y_1). If $y_1=0$, then C intersects C' at $(l,0)$; contradiction. Without loss of generality, let $y_1>0$. Since C passes through (l,y_1), C' follows a continuous path from $(l,0)$ to $(2l,y_1)$. Furthermore, this is the last intersection of C' with the line $x=2l$.
Now consider the path of C from its last intersection with the line $x=l$, (l,y_1), to its *first* intersection with the line $x=2l$, say $(2l,y_2)$. Since C cannot intersect C', it is bounded below by C', as well as the lines $x=l$ and $x=2l$. Therefore we must have $y_2>y_1$.



If at any point C intersects $x=2l$ below the value of y_1, then it must pass to the right of the point $(2l,y_1)$ in a clockwise direction. If such a point exists, let it be $(2l,y_3)$ and let the rightmost point of this section of C be (w,z). Since the curves do not intersect, C' is now bounded by the section of C and between $(2l,y_2)$ and $(2l,y_3)$, as well as the line $x=2l$ which C' no longer intersects. However, the section of C' from $(2l,y_1)$ to $(3l,y_3)$, is the translation of the section of C from (l,y_1) to $(2l,y_3)$. It has a total width of $w-l$, yet is bounded inside a region with width $w-2l$. This is a contradiction.

Hence C cannot intersect $x = 2l$ below y_1. Therefore, the last intersection of C and $x = 2l$ is above y_1, which is the last intersection of C and $x = l$.



By induction, the last intersection of C and the line $x = ml$ must have a greater y-value than the last intersection of C and $x = (m-1)l$, for $1 < m \leq n$. In particular, the last intersection of C and the line $x = nl = 1$ must have a positive y-value and cannot reach the point $B(1, 0)$. This is a contradiction. Therefore, if l is the reciprocal of a positive integer, C must intersect C' and there exists a chord of length l parallel to AB.

(2) We will present counterexamples to all other choices of l. If $l > 1$, then the straight line from $(0, 0)$ to $(1, 0)$ does not contain any chords of length l parallel to AB. Now consider $l < 1$. Since it is not the reciprocal of a positive integer, we can write $1/(n + 1) < l < 1/n$.
Consider the curve joining the following $2n + 2$ points in sequence:

$$(0,0), (n, -n), (n+1, 1), (2n+1, 1-n), (2n+2, 2), (3n+2, 2-n),$$
$$(3n+3, 3), (4n+3, 3-n), \ldots, (n^2+2n, 0).$$

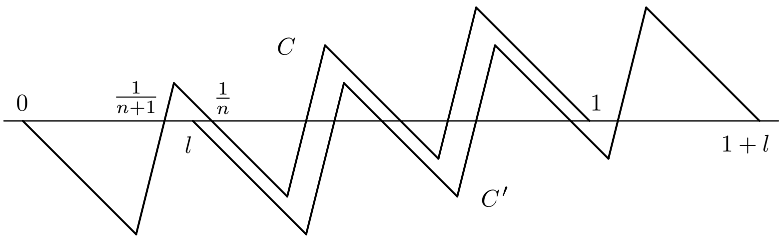
Other than $(0, 0)$, the next two points at which the curve intersects the x-axis are

$$\left(n + \frac{n}{n+1}, 0\right) = \left(\frac{n^2+2n}{n+1}, 0\right) \quad \text{and} \quad (n+1+1, 0) = (n+2, 0).$$

Now dilate the curve about $(0, 0)$ by a factor of $1/(n^2 + 2n)$. The result is a curve from $(0, 0)$ to $(1, 0)$ which intersects the x-axis at

$$\left(\frac{1}{n+1}, 0\right) \quad \text{and} \quad \left(\frac{1}{n}, 0\right).$$

This is the required curve C.



Translate C to the right by a distance of l to form C'. It is sufficient to check that C and C' do not intersect. This is true as C' is always below C. The first $2n-1$ line segments of C' is in fact the translation of the last $2n-1$ line segments of C, by the position vector of

$$\left(l - \frac{n+1}{n^2+2n}, -\frac{1}{n^2+2n}\right).$$

Therefore, C does not contain any chords of length l parallel to AB.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.