The gamma constant, $\gamma = 0.5772156\ldots$, is defined as

$$\gamma = \lim_{n \to \infty} (H(n) - \ln n)$$

$$= \lim_{n \to \infty} \left( \sum_{j=1}^{n} f(j) - \int_{1}^{n} f(x) \, dx \right),$$

where $f(x) = 1/x$ and $H(n)$ is the $n$th harmonic number

$$H(n) = \sum_{j=1}^{n} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$ 

This book poses certain problems to a reviewer for the *Gazette* such as the fact that it is an expository account and the readers of the *Gazette* are professionals—but that’s OK, as most of them, even those familiar with the gamma function, haven’t had much to do with this number. So I have not let the boast that the book has an introduction by the awesome Freeman Dyson intimidate me, as he would be quite impervious to any problems as to, say, notation or background that might trouble the general reader.

The Contents indicate that the author rambles over many areas, including the Benford distribution (nothing to do with gamma) and the Riemann hypothesis (only connected implicitly through the zeta function). This is acceptable, as many of us like to come up for some fresh air occasionally, look around and have a ramble after being confined to the rabbit hole in which we earn brownie points from our reductive offerings.

The extent of Havil’s rambling is also indicated in the Index which lists only a small number of pages actually devoted to the constant and, would you believe, in a book of 254 pages (apart from a throw-away on page 179), the last page that mentions gamma is page 159. Finally, it did not help to see that the first quotation is from Winston Churchill.

Apart from a one-liner in the Introduction, the definition of the constant is not given until page 47 and is preceded by a whole chapter on Napier as an introduction to logarithms. Napier’s logaithm is $Nap\, y = a \ln(a/y)$, where $a$ is an arbitrary constant. Napier’s approach also invokes a particle moving at a constant velocity as detailed in Tognetti [2, pp. 22, 23]. From this it is seen that his approach, although historically significant, represented a dead end from which it was necessary to back track to develop logarithms as we know them today. Furthermore it crops up again later in the book.
The result $\int_0^x (1/t) \, dt = \ln x$ is the fundamental relation from which the constant emerges and thus deserves a detailed proof. But the author has only mentioned in passing on page 13 that the logarithm might be related to the integral and then digresses into an historical comment. Havil states the result on page 47, along with a graphical demonstration that gamma is bounded on the unit interval. Surprisingly, he does not display the nice trick of horizontally translating the difference shapes into the unit square as he does with his note on the zeta function. However he does get a lower bound of 0.5 for the constant by using trapezia instead of rectangles — this turns out to be a very good estimate. But we don’t get any further estimates until page 69 where he goes to a lot of trouble to reproduce an existence proof that has an upper bound of about 1.65.

Despite these criticisms, Havil has quite nice chapters on the gamma, harmonic and zeta functions which prepare us for when he comes to the crunch as to why the gamma constant is so important: its underpinning of the number theory associated with the distribution of prime numbers. There are many references to this underpinning throughout the book and inclusion in the Index would have made it much easier to get an overall picture.

Having lost gamma along the way it was a welcome surprise to read Chapter 12 ‘Where is Gamma?’ And yes, at last, he helps us find it with more precision and makes some fascinating connections with other areas.

The book concludes in Chapter 16 with a large section on the Riemann hypothesis, replete with an appendix which is supposed to be a crash course in complex algebra (crash is the key word!).

I am delighted to say that, for me, Chapter 12 brings everything together, and reading it alone is enough to recommend that you buy the book. This book, and especially this chapter, has arguably more beautiful formulae than any other you will read, and if you were just to consider it as a gallery of such formula then the author has done a masterly job of displaying them, together with proofs — some of which are not so beautiful. The historical development, especially that of our Master Euler, is an added bonus. (In case you want to do a literature search, you should note that until recently, gamma was referred to as ‘Euler’s constant’, as in the case of [1], one of the best contemporary articles on gamma.)

The book should have been shorter and in three parts:

(i) a preliminary part developing the maths required for the level of exposition given in the other parts;
(ii) a succinct statement with proofs and refinements of the bounds for gamma; and
(iii) the connections with other areas and, in particular, why the constant is so important in the development of an understanding of prime number distribution.

The material on the Riemann hypothesis and other odds and ends such as the Benford distribution should have been the subject of a separate book.

References


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Leonhard Euler and the Bernoullis

M.B.W. Tent

M.B.W. Tent is a retired high-school maths teacher who has written a number of popular mathematical biographies aimed at students in Years 9 and 10 in (American) schools. Her latest is on the remarkable Swiss mathematicians who sprang from the region of Basel between 1650 and 1800.

Tent states explicitly that in order to make a coherent story, she has fabricated some details of their lives, and, in particular, their dialogues, which are presented as direct speech. The wisdom of this in a scholarly work is doubtful, but as she makes clear, this book has no such pretence and is instead an attempt to humanise the normally dry history of mathematics. I will therefore try to judge this book on the basis of what the author has tried to accomplish.

There were eight famous mathematical Bernoullis, all descendants of Nicolaus Bernoulli (1623–1708) and all called Jacob, Johann, Nicolaus or Daniel. Tent does a good job of distinguishing the first five, and describing their relationships, both professional and personal. She is also adept at describing the mathematics that was known at the time of their education, and the texts they probably studied. She is less successful in explaining their mathematical accomplishments, but, after all, how can you explain the calculus of variations, differential equations, the brachistochrone, the Petersburg paradox and so on to a student just beginning to study algebra, calculus and probability? Even elementary mathematical techniques, well within the capabilities of the intended audience, are not handled well. For example, Tent tries to explain the first Jacob’s attempt, as a schoolchild, to master the sum
of a finite geometric sequence. She gets to a point where a simple sentence, even without an explicit equation, could have completed the proof, but inexplicably breaks off there. I was also unconvinced by the purported reconstructions of the often acerbic conversations of the first Jacob (1654–1705) and his younger brother, the first Johann (1667–1748). We can deduce from their correspondence and publications how they probably interacted, but the conversations presented by Tent read more like the script of a television situation comedy.

One noteworthy feature of the careers of the mathematical Bernoullis is that they all preferred to live in their home town, Basel, on the outskirts of scholarly Europe, even when offered prestigious positions elsewhere. Unlike them, their friend and mathematical colleague, Leonhard Euler (1707–1783), also born in the vicinity of Basel, had no desire to return to his birthplace, even to visit his dying parents. As he is usually presented in biographies as a caring family man, the reason for this is unclear.

The second half of the book deals with Euler’s life and works. An extraordinarily productive polymath, Euler travelled widely in Europe, and was highly regarded among mathematicians and scientists. He had long, and often difficult, sojourns at the Academies of Peter the Great in Saint Petersburg and Frederick the Great in Berlin, and despite his family duties and eventual blindness, continued his mathematical and scientific researches and production of textbooks until the day he died. The author seems to be overwhelmed by the mathematical accomplishments of Euler. She emphasises several times how manifold they were, but for the reasons explained above, she fails to clearly explain any of them.

Tent’s book contains a number of trivial but annoying errors that should have been picked up by her editor. For example, Leibniz’ visit to London included conversations with members of the Royal Society, not the Royal Academy. The high end of Newton’s colour spectrum is red, not magenta. The fifth Fermat prime is 65,537, not 56,537. To sum up, this is an interesting read, well suited to its intended audience, but not a serious work of mathematical history.

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Pythagoras’ Revenge: A Mathematical Mystery

Arturo Sangalli

In recent years the genre of historical fiction has seen some boom times. *Pythagoras’ Revenge* is unlikely to achieve the runaway success of *The Da Vinci Code*, or the cult following of Bernard Cornwell’s hero Richard Sharpe, but it is a worthwhile and mathematically aligned offering. It combines elements of mystery, history and mathematics, with just a hint of the supernatural.

The central theme of the novel is that the famed Pythagoras had, contrary to popular belief, left a document penned by his own hand. The reader can well imagine the historical value of such a document, yet Sangalli is not content there. The contents of the document are alleged to be quite extraordinary, beyond mere mathematics, and so it turns out to be. This is the reason for the existence of the society that lurks in the shadows, willing to pay any price to recover the document. The search for this document, which is in two pieces, takes up most of the book.

The hunt introduces us to several characters, most of whom are well-intentioned people. There is no truly despicable character in this novel, although there are a few whose ethics have been overcome by their dreams. In fact, this is one of the weaknesses of the storyline—that there is insufficient conflict between the characters. There is, frankly, too much goodness for a genuinely enthralling tale. Nor is there enough action and suspense, especially for younger readers, who may well have grown up reading Matthew Reilly.

However, the novel does use the enjoyable literary technique of juxtaposing historical and contemporary narrative. Chapters touching on Pythagoras’ life and death, seen through the eyes of one of his students, are interspersed throughout the book. This is done in a sympathetic way, making the reader almost feel a part of the brotherhood. We learn more about the people involved, and the destruction of their group by outside forces. It was interesting to learn more about the Pythagoreans and their habits, but I would have liked to have seen a bit more mathematics in these sections. Of course, Sangalli has tried to make the whole book as accessible as possible to mathematically unsophisticated readers, and has done a fine job.

There is some mathematics, particularly in the early chapters where odd and even permutations are discussed. Later there is some discussion of commensurable and incommensurable quantities, with a segue to how irrational numbers would have solved the Pythagoreans’ well-known problem with the square
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root of 2. This puts both the ancient and modern viewpoints on this issue quite well. It also serves as a good motivation for the introduction of the mathematical genius the secret society has been waiting for. His work is summarised quite nicely, and is based on historical work by Turing and Chaitin. Unfortunately for him, the society will stop at nothing to get him into its clutches. The twist in this story, and it is a good one, is how modern mathematics relates to the ancient, and how it resolves the dreams of the society members.

There are several appendices discussing mathematical concepts in lay language. These could provide some interesting seed material for student projects. The historical information appears largely accurate, and the insights into fields such as archaeology and historical research also look reliable. I was particularly amused by the historical insight into Sam Loyd, the famous puzzler. But in the end Pythagoras’ Revenge is a novel, and should be read that way. Was it a page turner? No, but it deserves to be widely read, and it should certainly appear in every high-school library. Hopefully it will receive the circulation it deserves, and perhaps even spur the author (or others) on to future efforts. There is certainly no shortage of material for the imagination in mathematical history—Archimedes, Newton, Galois, Hypatia, anyone?

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