Pricing and risk measurement with backward stochastic differential equations

Samuel N. Cohen*

Arguably the key question in much of mathematical finance is, 'How does one measure the worth of an uncertain outcome?' While much work has been done on this over the centuries, from Daniel Bernoulli through to von Neumann–Morgenstern, many problems in financial markets require new approaches.

Recent work has focussed on the development of *nonlinear expectations*, and the related *dynamic risk measures*. These are operators taking random outcomes known at a future time and assigning them values at previous times (formally, for a probability space with filtration $\{\mathcal{F}_t\}$, they are maps $L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t)$; $t \leq T$). Nonlinear expectations generalise the classical notion of conditional expectation, and should obey basic axioms (e.g. monotonicity and time-consistency), but allow risk aversion in preferences to be directly modelled.

Of course, while it is easy to specify a set of axioms for these nonlinear expectations, it is nontrivial to construct interesting examples. A tool which is commonly used is the theory of backward stochastic differential equations (BSDEs). These are stochastic differential equations where a stochastic terminal condition is given, rather than an initial condition. The value of this terminal condition is generally unknown today, and will not be revealed until a future time T. Nevertheless, a solution to such an equation should be adapted, that is, should not depend on knowledge of the future. To deal with this, we allow the solution to consist of two parts — a 'value' Y and a 'control' Z, where Z allows hedging of the stochastic part of the outcome. It is then possible to show (see [1]) that such equations have unique solutions, for arbitrary (finite variance) terminal conditions, where the dynamics may depend on Y and Z in a nonlinear way.

In [2] and [3], we explore problems in the theory of BSDEs, where randomness does not arise from a standard Brownian motion. In [2], we consider BSDEs where randomness arises from a continuous-time finite-state Markov chain, and derive a comparison theorem, which corresponds to a stochastic version of Pontryagin's maximum principle in this context. In [3], we consider the corresponding equations in a discrete-time finite-state world, and derive existence and uniqueness results,

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Samuel Cohen was awarded the T.M. Cherry Prize for best student presentation at ANZIAM 2009, and the B.H. Neumann Prize for best student presentation at AustMS 2009. This extended abstract is an invited contribution to the *Gazette*.

*Applied Mathematics, The University of Adelaide, North Terrace, Adelaide, 5005.
Email: samuel.cohen@adelaide.edu.au

along with an appropriate version of the comparison theorem. Using this theory, we can construct nontrivial examples of nonlinear expectations in these spaces, and in discrete time, show that *every* nonlinear expectation must be the solution to a BSDE.

References

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Samuel attended high school in Hamilton, Victoria, and then proceeded to the University of Adelaide, where he studied mathematics and finance. In 2007 he studied for Honours in Statistics, and then proceeded to a PhD under the supervision of Robert Elliott and Charles Pearce. His main mathematical interests are in probability theory and its interaction with economics and finance. In October, he will take up a position as a Junior Research Fellow at St John's College, Oxford.