



Book reviews

Topics in mathematical modelling

K.K. Tung

Princeton University Press, 2007, ISBN-13: 978-0691116426

We think that many students whose interests are mainly in applications have difficulty in following abstract arguments, not on account of incapacity, but because they need to ‘see the point’ before their interest can be aroused.

Jeffreys and Jeffreys [7]

From a community wide perspective, mathematical modelling is the new emerging image of and expectation for mathematics.

The comprehensive availability of inexpensive and powerful computing, the ease with which knowledge can be quickly accessed using search engines, such as Web of Science and Google, and the speed with which symbolic and algebraic calculations can be performed using packages, such as Maple, Matlab and Mathematica, are having a profound effect on how mathematics is now viewed, performed and utilised. In many endeavours, through the use of an appropriate mathematical model, computational experimentation has become the less expensive alternative to direct experimentation. This is most apparent in industrial and biological applications. By using the Navier–Stokes equation as the appropriate mathematical model, wind tunnel experiments now play quite minor roles in the design of new aircraft. The Boltzmann causal integral equation models of linear viscoelasticity play a central role in the analysis and interpretation of the stress–strain response of biological cells [3] to controlled deformations.

In this book, one finds a variety of examples of the use of mathematical models as an alternative to direct experimentation. Chapter 4 discusses the role of first-order differential equation models in carbon dating, the estimation of the age of the Universe and HIV modelling. Under the heading ‘Interactions’, Chapter 9 surveys predator–prey dynamics and the control of pests by spraying, and hypothesises about the lack of large carnivores in Australia today.

The title of the book aptly describes its essential content: various topics about the application of mathematics to practical real-world problems. It contains a comprehensive variety of independent examples of mathematical modelling. Its appeal is the novelty of the various choices and the innovative way in which they have been presented.

As a direct result of this increasing importance of mathematical modelling in everyday activities, as well as other factors, the way that we do science is changing [2]. There is a need to take such matters into account in terms of how mathematics is communicated. The future prospects for the mathematics profession hinges on how this is achieved. The community’s perception and response will depend heavily on how the mathematics profession makes the practice of mathematical modelling accessible to and stimulating for physical and social scientists, engineers, financial and managerial professionals, teachers and students. The philosophy and

methodology of mathematical modelling must be marketed, communicated and taught in innovative, interesting and enthusiastic ways. For achieving this, an appropriate rationale and strategy would be to respond to the advice implicit in the above comment by Jeffreys and Jeffreys — assist the student to see the point.

Through interesting historical facts and novel details, included in each chapter, to stimulate the interest of the reader as well as present a comprehensive background to the modelling involved, the present book goes a long way to accommodating that implicit advice. In addition, for the audience for whom it has been written, this book is a very good introduction to mathematical modelling for the following reasons:

- The breadth of the applications covered in the different chapters: phyllotaxis; scaling laws; HIV modelling; carnivores in Australia; marriage and divorce; El Niño and the southern oscillation index; collapsing bridges. In this way, it highlights the role of mathematical modelling in everyday activities: HIV modelling; marriage and divorce; El Niño.
- It illustrates that mathematical modelling, as well as a mathematical endeavour, is a thoughtful problem-solving process where the relevant background and insight comes from a variety of sources.
- Its use of a common structure for each chapter, which involves painting the big picture in which the modelling problem sits through the inclusion of interesting historical facts and novel insights.
- The care and thought with which it has been put together. The simple models discussed illustrate a key aspect of mathematical modelling: use no more mathematical sophistication than is necessary to answer the question under examination.

There is no unique way to teach mathematical modelling. There are a variety of good books depending on the circumstances. They include:

- Fowkes and Mahony's [4] book is for the teaching of the dedicated mathematics student who wants to be able to participate in challenging mathematical modelling endeavours with skill and confidence. The emphasis is strongly motivated by mathematical modelling that has arisen in industrial situations, and stresses how the same mathematical model can arise in entirely different contexts.
- Harte's two books, *Consider a Spherical Cow* and *Consider a Cylindrical Cow* ([5], [6]), are for learning about the essence of mathematical modelling at a more philosophical and intuitive level, where environmental science problem-solving is the principal focus.
- The current book is for the advanced undergraduate student who is already familiar with the concepts of ordinary differential equations etc. and who wants mathematical modelling to be a skill that complements and supplements their chosen professional expertise.

No matter who we are, we are doing mathematical modelling in one way or another. At the level of our basic mathematical knowledge, we do it with skill and confidence and take it for granted. For some, it is planning a holiday, for others it is their tax return, for the engineer it is often the repeating of a standard calculation with different inputs, etc. The approach taken in this book will stimulate the awareness of students to this fact.

No book is perfect, even if, from some perspective, it is a very good book. For the current text, some oversights include:

- The birth–death dates for Henri Poincarè, on p. 213, should be 1854–1912 not 1647–1727.
- The index is not sufficiently comprehensive given the purpose of the book (e.g. Poincarè is mentioned in the contents, but not listed in the index).
- There is an over-emphasis on the role of first-order ordinary differential equations.
- The importance for mathematical modelling of symbolic manipulation packages such as Maple, Mathematica and Matlab are not sufficiently strongly stressed.
- The lack of an initial overview introductory chapter that explains the philosophy and rationale of mathematical modelling as a strategy for action. Some relevant remarks are included in the preface, but are insufficient.

There are some strongly differing views on how introductory mathematics and mathematical modelling should be linked and taught. The views range between the two extremes of

- (a) having skill and confidence with the mathematics that will arise in the mathematical modelling to be taught, and
- (b) using mathematical modelling to discover new mathematics.

The current book opts for the former approach. For the student learning mathematics to complement and supplement some chosen profession (engineer, medical professional, scientist, sociologist etc.) this is the appropriate strategy [1]. The latter approach should be reserved for the mathematically talented.

In summary, this is a good introductory book about the nature and purpose of mathematical modelling. The topics chosen and the way in which they have been motivated and presented will help a wide range of students to ‘see the point’ and thereby arouse and stimulate their confidence about their mathematical problem solving skills.

References

- [1] Broadbridge, P. (2007). *AMSI News. Gaz. Aust. Math. Soc.* **34**, 230–232.
- [2] de Hoog, F.R. (2007). Maths matters: opportunities for the mathematical sciences? *Gaz. Aust. Math. Soc.* **34**, 137–142.
- [3] Desprat, N., Richert, A., Simeon, J. and Asnacios, A. (2005). Creep function of a living cell. *Biophys. J.* **88**, 2224–2233.
- [4] Fowkes, N.D. and Mahony, J.J. (1994). *An Introduction to Mathematical Modelling*. John Wiley and Sons, Chichester.
- [5] Harte, J. (1988). *Consider a Spherical Cow*. University Science Books, Sausalito, CA, USA.
- [6] Harte, J. (2001). *Consider a Cylindrical Cow*. University Science Books, Sausalito, CA, USA.
- [7] Jeffreys, H. and Jeffreys, J. (1946). *Methods of Mathematical Physics*. Cambridge University Press.

Bob Anderssen

CSIRO Mathematical and Information Sciences, GPO Box 664, Canberra, ACT 2601.

E-mail: Bob.Anderssen@csiro.au



A course in calculus and real analysis

S.R. Ghorpade and B.V. Limaye
Springer, 2006, ISBN 978-0-387-30530-1

The important new text *A Course in Calculus and Real Analysis* by S.R. Ghorpade and B.V. Limaye (Springer UTM) is a rigorous, well-presented and original introduction to the core of undergraduate mathematics — *first year calculus*. It develops this subject carefully from a foundation of high-school algebra, with interesting improvements and insights rarely found in other books. Its intended audience includes mathematics majors who have already taken some calculus, and now wish to understand the subject more carefully and deeply, as well as those who teach calculus at any level. Because of the high standard, only very motivated and capable students can expect to learn the subject for the first time using this text, which is comparable to Spivak's *Calculus* [1], or perhaps Rudin's *Principles of Mathematical Analysis* [2].

The book strives to be precise yet informative at all times, even in traditional 'hand waving' areas, and strikes a good balance between theory and applications for a mathematics major. It has a voluminous and interesting collection of exercises, conveniently divided into two groups. The first group is more routine, but still often challenging, and the second group is more theoretical and adds considerable detail to the coverage. However no solutions are presented. There are a goodly number of figures, and each chapter has an informative Notes and Comments section that makes historical points or otherwise illuminates the material.

The authors based the book on some earlier printed teaching notes and then spent seven years putting it all together. The extensive attention to detail shows, and an honest comparison with the calculus notes generally used in Australian universities would be a humbling exercise. Here is a brief indication of the contents of the book by chapter.

Chapter 1 (Numbers and Functions) introduces integers, rational numbers and the basic properties of real numbers, without defining exactly what real numbers are — this difficulty seems unavoidable in an elementary text. Inequalities and basic facts about functions are introduced, and the main examples are *polynomials* and their quotients, the *rational functions*. This book is notable for not using the log, exponential and circular functions until they are properly defined: these appear roughly half-way through the book.

Boundedness, *convexity* and *local extrema* of functions are defined, and a function $f(x)$ is said to have the *Intermediate Value Property* (IVP) if r between $f(a)$ and $f(b)$ implies $r = f(x)$ for some x in $[a, b]$. The geometric nature of these notions is thus brought to the fore, before the corresponding criteria for them in terms of continuity and differentiability are introduced.

Chapter 2 (Sequences) introduces the basic idea of convergence used in the text: a sequence a_n of real numbers *converges to* a if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq n_0$. Convergence of bounded monotonic sequences is shown, and the *little-oh* and *big-oh* notations are briefly introduced. Every real sequence is shown to have a monotonic subsequence, and from this follows the Bolzano–Weierstrass theorem that every bounded sequence has a convergent subsequence. Convergent sequences are shown to be Cauchy and conversely.

In Chapter 3 (Continuity and Limits), the book strikes out into less charted waters. A function $f(x)$ defined on a domain D , which is allowed to be an arbitrary subset of \mathbb{R} , is *continuous at* $c \in D$ if for any sequence x_n in D converging to c , $f(x_n)$ converges to $f(c)$. So continuity, defined in terms of sequences, occurs before the notion of the limit of a function, and works for more general domains than the usual setup. This approach is probably more intuitive for students, and has been used in other texts, for example, Goffman's *Introduction to Real Analysis* [3].

The usual $\varepsilon - \delta$ formulation is shown to be equivalent to the above condition. A strictly monotonic function f defined on an interval I has an inverse function f^{-1} defined on $f(I)$ which is continuous. A continuous function defined on an interval is shown to have the IVP. Uniform continuity is also defined in terms of sequences: if f is defined on D , then it is *uniformly continuous* on D if x_n and y_n sequences in D with $x_n - y_n \rightarrow 0$ implies that $f(x_n) - f(y_n) \rightarrow 0$. If D is closed and bounded then a continuous function on D is shown to be uniformly continuous on D . If D is a set which contains open intervals around c then a function f defined on D has *limit* l as x approaches c if for any sequence x_n in $D \setminus \{c\}$ converging to c , the sequence $f(x_n)$ converges to l . So again the notion of the limit of a function comes down to the concept of limits of sequences. The usual $\varepsilon - \delta$ formulation is shown to be equivalent to this definition. Relative notions of little-oh and big-oh between two functions as $x \rightarrow \infty$ are introduced, and more generally there is a careful discussion of infinite limits of functions and *asymptotes*, including oblique asymptotes.

Chapter 4 (Differentiation) introduces the *derivative* of a function $f(x)$ at a point c as the usual limit of a quotient. Then it proves the Lemma of Carathéodory, which becomes crucial in what follows: that f is differentiable at c if and only if there is an *increment function* f_1 such that $f(x) - f(c) = (x - c)f_1(x)$ for all x in the domain D of f , and f_1 is continuous at c . By replacing differentiability of f at c with the continuity of f_1 (which depends on c) at c , routine properties of derivatives — continuity, sums, products, quotients and especially the Chain rule and the derivative of an inverse function — have more direct proofs which no longer require mention of limits.

There is then a discussion of normals and implicit differentiation and the *Mean Value Theorem* (MVT), which is used to prove Taylor's theorem: we can express an n -times differentiable function $f(x)$ on $[a, b]$ by an n degree *Taylor polynomial* with an error term involving the $(n + 1)$ th derivative at some interior point c . The connection between derivatives and monotonicity, convexity and concavity are discussed, and the chapter ends with an unusually careful and thorough treatment of L'Hôpital's rule, treating both $0/0$ and ∞/∞ forms with some care.

Chapter 5 (Applications of Differentiation) begins with a discussion of *maxima* and *minima*, *local extrema* and *inflection points*. Then the *linear* and *quadratic approximations* to a function f at a point c given by Taylor's theorem are studied in more detail, including explicit bounds on the errors as x approaches c . The most novel parts of this chapter are a thorough treatment of *Picard's method* for finding a fixed point of a function $f : [a, b] \rightarrow [a, b]$ provided $|f'(x)| < 1$, and *Newton's method* for finding the zeros of a function $f(x)$. Conditions are given that insure that the latter converges, one such condition uses Picard's method, the other assumes the monotonicity of $f'(x)$.

Chapter 6 (Integration) is the heart of the subject. Many calculus texts introduce the integral of a function as some kind of ‘limit of Riemann sums’, even though this kind of limit has not been defined, as it ranges over a net of partitions, not a set of numbers. Ghorpade and Limaye choose another standard approach: to define the *Riemann integral* using supremums and infimums of sets of real numbers. Given $f(x)$ on $[a, b]$, they define the lower sum $L(P, f)$ and the upper sum $U(P, f)$ of f with respect to a partition P of $[a, b]$ in terms of minima and maxima of f on the various subintervals, then set

$$L(f) \equiv \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$$

$$U(f) \equiv \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$$

and declare f to be *integrable* on $[a, b]$ if $L(f) = U(f)$, in which case this common value is the definite integral $\int_a^b f(x) dx$. This is a definition which is reasonably intuitive, and respects Archimedes’ understanding that one ought to estimate an area from both the inside and outside to get proper control of it. Nevertheless one must make the point that no good examples of using this definition to compute an integral are given — while the book shows that $f(x) = x^n$ is integrable, an evaluation of the integral must wait for the Fundamental theorem.

A key technical tool is the *Riemann condition*: that a bounded function is integrable if and only if for any $\varepsilon > 0$ there is a partition P for which the difference between the lower and upper sums is less than ε . The *Fundamental theorem* is established, in both forms: that the integral of a function may be found by evaluating an antiderivative, and that the indefinite integral of a function f is differentiable and has derivative f . Integration by parts and substitution are derived, and then the idea of a Riemann sum is introduced both as a tool to evaluate integrals, and to allow integration theory to evaluate certain series.

Chapter 7 (Elementary Transcendental Functions) introduces the *logarithm*, the *exponential function*, and the *circular functions* and their *inverses*. The book defines $\ln x$ as the integral of $1/x$ and the exponential function as its inverse, and develops more general power functions using the exponential function and the log function. The number e is defined by the condition $\ln e = 1$. This is familiar territory. Defining $\sin x$, $\cos x$ and $\tan x$ is less familiar, but a crucial point for calculus. Most texts are sadly lacking, pretending that these functions are somehow part of the background ‘ether’ of mathematical understanding, and so exempt from requiring proper definitions. More than fifty years ago, G.H. Hardy spelled out the problem quite clearly in his *A Course in Pure Mathematics* [4], stating ‘The whole difficulty lies in the question, what is the x which occurs in $\cos x$ and $\sin x$ ’. He described four different approaches to the definition of the circular functions.

The one taken by Ghorpade and Limaye is to start with an *inverse circular function*. There are several good reasons to justify this choice. Historically the inverse circular functions were understood analytically before the circular functions themselves; Newton obtained the power series for $\sin x$ by first finding the power series for $\arcsin x$ and then inverting it, and indeed the $\arcsin x$ series was discovered several centuries earlier by Indian mathematicians in Kerala. In addition, the theory of elliptic functions is arguably easier to understand if it proceeds by analogy with the circular functions, and starts with the inverse functions — the elliptic integrals.

The book begins with $\arctan x$, the integral of $1/(1+x^2)$ which, after $1/x$, is the last serious barrier to integrating general rational functions. Defining $\tan x$ as the inverse of $\arctan x$ only defines it in the range $(-\pi/2, \pi/2)$, where π is introduced as twice the supremum of the values of $\int_0^a 1/(1+x^2) dx$. Then the circular functions

$$\sin x = \frac{\tan x}{\sqrt{1 + \tan^2 x}} \quad \text{and} \quad \cos x = \frac{1}{\sqrt{1 + \tan^2 x}}$$

are defined on $(-\pi/2, \pi/2)$ as suggested by Hardy, extended by continuity to the closed interval, and then to all of \mathbb{R} by the rules

$$\sin(x + \pi) = -\sin x \quad \text{and} \quad \cos(x + \pi) = -\cos x.$$

I would suggest an alternative: to define

$$\sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)} \quad \text{and} \quad \cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

on $(-\pi, \pi)$ and then extend by continuity and 2π periodicity. This more algebraic approach connects to the rational parametrisation of the circle, Pythagorean triples, and the well-known half-angle substitution.

After $\sin x$ and $\cos x$ are pinned down, the book finds their n th Taylor polynomials and derives the main algebraic relations, namely $\cos^2 x + \sin^2 x = 1$ and the addition laws. After defining the *reciprocal* and the (other) *inverse circular functions* the book establishes their derivatives. It then discusses a good source of counterexamples: the function $\sin(1/x)$.

Having defined the circular functions precisely, the authors define the *polar coordinates* r and θ of a point $(x, y) \neq (0, 0)$ in the Cartesian plane precisely: $r = \sqrt{x^2 + y^2}$ as usual, while

$$\theta = \begin{cases} \cos^{-1}\left(\frac{x}{r}\right) & \text{if } y \geq 0 \\ -\cos^{-1}\left(\frac{x}{r}\right) & \text{if } y < 0. \end{cases}$$

References

- [1] Spivak, M. (1994). *Calculus*, 3rd edn. Publish or Perish.
- [2] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd edn. McGraw-Hill.
- [3] Goffman, C. (1966). *Introduction to Real Analysis*. Harper and Row, New York.
- [4] Hardy, G.H. (1908). *A Course of Pure Mathematics*, 1st edn. Cambridge University Press. Free electronic version of the 1908 edition available at <http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=ACM1516.0001.001> (accessed 18 October 2007).

N.J. Wildberger

School of Mathematics and Statistics, UNSW, Sydney 2052. E-mail: n.wildberger@unsw.edu.au

◇ ◇ ◇ ◇ ◇ ◇

A madman dreams of Turing machines

Janna Levin

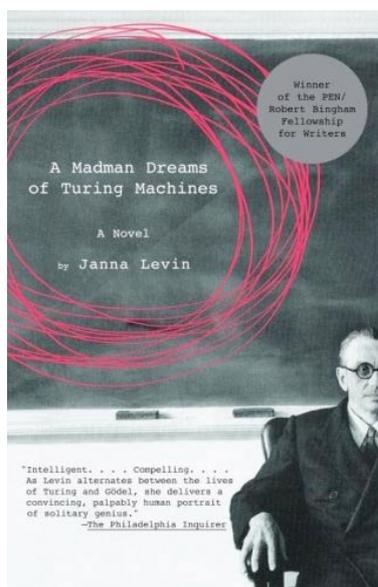
Anchor Books, 2007, ISBN 978-1-4000-3240-2

A Madman Dreams of Turing Machines by Janna Levin is a fictionalised biography of Alan Turing and Kurt Goedel. The book takes us from Turing's days as a school boy at Sherbourne School to his death in 1954, and from Goedel's life in Vienna at around the time he proved his Incompleteness Theorems, to his death in the US in 1978.

The lives of both men were tragic. The main facts of Turing's life — his involvement with the war-time code breaking at Bletchley Park, his being charged with homosexual practices and his suicide by eating a poisoned apple — are well known and have even become a part of popular culture. But what came as news (to me, at least) were the circumstances leading to Goedel's death. As portrayed by Levin, from his early manhood Goedel displayed an unusual degree of concern about possible poisons in his environment. He was worried that burning coal gave off harmful fumes. As he got older, these worries deepened into paranoia. He became convinced people were trying to poison him. Eventually, he stopped eating altogether and died of self-inflicted starvation. Turing died from deliberately eating poison, Goedel from not eating due to a fear of poison.

Levin's book interweaves the lives of the two mathematicians. She spends a few chapters, or even just a few pages, on one of them and then switches to the other. But I never found this to be at all confusing or disturbing: it gives an impression of the two lives progressing in parallel.

Although Levin does sketch, in very broad outline, some of their mathematical ideas, this is not a book about mathematics. It is a book about their personal lives, but also about the broader intellectual and cultural environment in which their thought developed. Goedel was a member of the Vienna Circle, a passionately anti-metaphysical group of thinkers centred around Moritz Schlick. Goedel's Incompleteness Theorems, and his Mathematical Platonism, were not congenial to the general intellectual mood of this group. Later in his life Goedel developed a variant of the Ontological Argument for the existence of God and, according to Levin, also believed in the immortality of the soul. Turing, by contrast, was a dyed-in-the-wool materialist who firmly believed that human beings were nothing but machines. But, oddly enough, there is also a letter written by Turing in which he expressed the view that an early school boy love who had died of tuberculosis was in some sense still alive.



As portrayed by Levin, both men had very unusual personalities. Although in his early years Goedel was something of a dandy and a ladies' man, as he got older he became increasingly difficult and paranoid. He had to be spoon fed by his wife and would wear many layers of clothing even in warm weather. Turing is portrayed as a strange man, indifferent to personal grooming and almost blind to the emotional sub-text in social interactions.

One thing that I (as a philosopher) found surprising was the extent of the (not always entirely benign) influence that Ludwig Wittgenstein had on both men. Wittgenstein's ideas dominated the Vienna Circle. Turing attended Wittgenstein's lectures at Cambridge, and Levin gives a detailed account of their exchanges.

It seems to me that Levin's book is very well written. The prose is finely polished. She sometimes achieves a strange, dream-like quality in her writing in which the distinction between past and present somehow becomes unimportant, and events in the natural world seem to take on something of the 'timeless' quality of mathematics. She has a fine feeling for language. It is worth noting that Levin is a Professor of Physics and Astronomy at Columbia University, and is also an accomplished visual artist and award winning novelist(!).

This is an impressive, in some ways beautiful, but also very sad book about two great thinkers. I strongly recommend it.

J.N. Wright

School of Humanities and Social Science, The University of Newcastle, University Drive,
Callaghan, NSW 2308. E-mail: jnwright@westnet.com.au

