pp. 299–315: D. Caponetti, A. Trombetta and G. Trombetta
Proper 1-ball contractive retractions in Banach spaces of measurable functions.
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PROPER 1-BALL CONTRACTIVE RETRACTIONS IN BANACH SPACES OF MEASURABLE FUNCTIONS

D. Caponetti, A. Trombetta and G. Trombetta

In this paper we consider the Wośko problem of evaluating, in an infinite-dimensional Banach space $X$, the infimum of all $k \geq 1$ for which there exists a $k$-ball contractive retraction of the unit ball onto its boundary. We prove that in some classical Banach spaces the best possible value 1 is attained. Moreover we give estimates of the lower H-measure of noncompactness of the retractions we construct.

1. Introduction

Let $X$ be an infinite-dimensional Banach space with unit closed ball $B(X)$ and unit sphere $S(X)$. It is well known that, in this setting, there is a retraction of $B(X)$ onto $S(X)$, that is, a continuous mapping $R : B(X) \to S(X)$ with $Rx = x$ for all $x \in S(X)$. In [4] Benyamini and Sternfeld, following Nowak ([13]), proved that such a retraction can be chosen among Lipschitz mappings. The problem of evaluating the infimum $k_0(X)$ of the Lipschitz constants of such retractions is of considerable interest in the literature. A general result states that in any Banach space $X$, $3 \leq k_0(X) \leq k_0$ (see [8, 10]), where $k_0$ is a universal constant. In special spaces more precise estimates have been obtained by means of constructions which depend on each space. We refer the reader to [9, 10] for a collection of results on this problem and related ones.

A similar problem can be considered by replacing Lipschitz retractions by $k$-ball contractive retractions. Let us recall that for a bounded $A \subset X$, the Hausdorff measure (briefly $H$-measure) of noncompactness $\gamma(A)$ is the infimum of all $\varepsilon > 0$ such that $A$ has a finite $\varepsilon$-net in $X$. The following properties of $\gamma$ hold, for bounded $A, B \subset X$:

\[
\begin{align*}
\gamma(A) &= 0 \text{ if and only if } A \text{ is precompact;} \\
\gamma(\overline{\sigma A}) &= \gamma(A) \text{ where } \overline{\sigma A} \text{ denotes the closed convex hull of } A; \\
\gamma(A \cup B) &= \max\{\gamma(A), \gamma(B)\}; \\
\gamma(A + B) &\leq \gamma(A) + \gamma(B); \\
\gamma(\lambda A) &= |\lambda| \gamma(A), \text{ for all } \lambda \in \mathbb{R}.
\end{align*}
\]

Received 5th May, 2005

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A continuous mapping $T : \text{dom}(T) \subset X \to X$ is called \textit{k-ball contractive} if there is $k \geq 0$ such that $\gamma(TA) \leq k \gamma(A)$ for each bounded $A \subset \text{dom}(T)$.

In [20] Wośko has proved that in the space $X = C([0,1])$ for any $\varepsilon > 0$ there exists a $(1 + \varepsilon)$-ball contractive retraction of $B(X)$ onto $S(X)$. Moreover he has posed the question of estimating the characteristic:

$$W(X) = \inf \{ k \geq 1 : \text{there is a } k\text{-ball contractive retraction } R : B(X) \to S(X) \}$$

for special classical Banach spaces, and also the question whether or not there is a Banach space in which $W(X)$ is a minimum. As Wośko has pointed out a 1-ball contractive retraction cannot be a Lipschitz mapping. In [19] it was shown that $W(X) \leq 6$ for any Banach space, reaching the value 4 and 3 depending on the geometry of the space $X$. Results in other Banach spaces can be found in [6, 12, 16, 17]. Recently, in [1, Theorem 4] it has been proved that if the Banach space $X$ has a monotone norm, then for any $\varepsilon > 0$ there exists a $(1 + \varepsilon)$-ball contractive retraction of $B(X)$ onto $S(X)$. For a continuous mapping $T : \text{dom}(T) \subset X \to X$ we also consider the following quantitative characteristic which is of interest in nonlinear analysis:

$$\omega(T) = \sup \{ k \geq 0 : \gamma(TA) \geq k \gamma(A) \text{ for every bounded } A \subset \text{dom}(T) \}$$

called the \textit{lower H-measure of noncompactness} of $T$. This characteristic is closed related to properness. In fact, from $\omega(T) > 0$ it follows that $T$ is a \textit{proper} mapping, that is, $T^{-1}K$ is compact for each compact subset $K$ of $X$.

Aim of this paper is to estimate $W(X)$ in some classical Banach spaces of real valued measurable functions on $[0,1]$ and also to give estimates of the lower H-measure of noncompactness of the retractions we construct. In Section 3 we consider special Banach spaces in which, by means of a suitable compact mapping $P_X : B(X) \to X$, we give an explicit formula of a $k$-ball contractive retraction with positive lower H-measure of noncompactness. In the sections which follow we give examples of Banach spaces $X$ in which $W(X) = 1$. In Orlicz (Section 4) and Lorentz spaces (Section 5) we obtain that the value $W(X) = 1$ is actually a minimum. Moreover in Lebesgue and Lorentz spaces we show that a 1-ball contractive retraction $R$ can be chosen in such a way that $\omega(R) = 1$. As a consequence in the Lebesgue and Lorentz spaces we have the existence of 1-ball contractive fixed point free mappings $F : B(X) \to B(X)$ with $\omega(F) = 1$.

2. Preliminaries.

Let $\Sigma$ be the $\sigma$-algebra of all Lebesgue measurable subsets of $[0,1]$ equipped with the Lebesgue measure $\mu$, and write \textit{almost everywhere} for $\mu$-almost everywhere. Let $\mathcal{M}_0 := \mathcal{M}_0([0,1],\Sigma,\mu)$ denote the space of all classes of Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ and $\mathcal{M}_0^+$ its positive cone. We recall the definition of Banach function space, we refer to the book of Bennett–Sharpley [3] for the main results of this theory.
Definition 2.1. A mapping $\rho : \mathcal{M}_0^+ \rightarrow [0, \infty]$ is called a Banach function norm if, for all $f, g, f_n (n = 1, 2, \ldots)$ in $\mathcal{M}_0^+$, for all constants $\lambda \geq 0$ and for all $E \in \Sigma$, the following properties hold:

$\rho(f) = 0$ if and only if $f = 0$ almost everywhere in $[0, 1]$;
$\rho(\lambda f) = \lambda \rho(f)$;
$\rho(f + g) \leq \rho(f) + \rho(g)$;
$g \leq f$ almost everywhere $\Rightarrow \rho(g) \leq \rho(f)$;
$f_n \uparrow f$ almost everywhere $\Rightarrow \rho(f_n) \uparrow \rho(f)$;
$\rho(\chi_{[0,1]}) < \infty$;
$\int_E f(t) \, dt \leq C_E \rho(f)$, for some constant $0 < C_E < \infty$ independent of $f$.

Definition 2.2. If $\rho$ is a Banach function norm, the Banach space

$Y = \left\{ f \in \mathcal{M}_0 : \rho(|f|) < \infty \right\}$

is a Banach function space, endowed with the norm $\|f\| = \rho(|f|)$.

Throughout this section $Y$ is a Banach function space.

Definition 2.3. A function $f \in Y$ is said to have absolutely continuous norm if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f \chi_D\| < \varepsilon$ for every $D \in \Sigma$ with $\mu(D) < \delta$.

Note that, as the underlying space $[0, 1]$ has finite measure, by virtue of [18, Lemma 3.3.2], the above definition is equivalent to [3, Definition 3.1]. We set

$Y^0 = \{f \in Y : f$ has absolutely continuous norm$\}$.

If $Y^0 = Y$, then the space $Y$ is said to have absolutely continuous norm. We denote by $W$ the set of all simple functions of $\mathcal{M}_0$. We recall that $W$ is a subset of $Y$ and we denote by $\overline{W}_{\|\cdot\|}$ the closure of $W$ in $Y$.

The next lemma collects some results we need (see [3, Theorems 3.8, 3.11 and 3.13]).

Lemma 2.4. The following statements hold:

(i) The space $Y^0$ is an order ideal of $Y$, that is, it is a closed linear subspace of $Y$ with the property:

\[ f \in Y^0 \text{ and } |g| \leq |f| \text{ almost everywhere } \Rightarrow g \in Y^0. \]

(ii) The subspace $\overline{W}_{\|\cdot\|}$ is an order ideal of $Y$ and $Y^0 \subset \overline{W}_{\|\cdot\|} \subset Y$.

(iii) The subspaces $Y^0$ and $\overline{W}_{\|\cdot\|}$ coincide if and only if the characteristic function $\chi_{[0,1]}$ has absolutely continuous norm. In particular, $Y^0 = \overline{W}_{\|\cdot\|} = Y$ whenever $Y$ has absolutely continuous norm.

We recall the following useful characterisation of convergent sequences in $Y^0$. 

Lemma 2.5. ([2, p. 41]) A sequence \( \{f_n\} \) converges to \( f \) in \( Y^0 \) if and only if \( \{f_n\} \) converges to \( f \) in measure and the family \( \{f_n : n \in N\} \) has uniformly absolutely continuous norm, that is, for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \sup_n \|f_n \chi_D\| < \varepsilon \) for every \( D \in \Sigma \) with \( \mu(D) < \delta \).

Let \( C([0,1]) \) denote the Banach space of all real and continuous functions on \([0,1]\) endowed with the sup norm \( \|\cdot\|_\infty \). By a standard argument (see for example [15, Theorem 3.14]) it can be shown the following lemma.

Lemma 2.6. Assume \( Y^0 = \overline{W}^{\|\cdot\|} \), then \( C([0,1]) \) is dense in \( Y^0 \).

3. Proper \( k \)-ball contractive retractions: abstract results.

Let \( X \) denote the Banach space of all functions of absolutely continuous norm of a Banach function space \( Y \). We still denote by \( W \) the subset of \( Y \) of all simple functions. For \( f \in X \) and \( a \in [1,2] \), we set

\[
f_a(t) = \begin{cases} f(at) & \text{if } t \in \left[0, \frac{1}{a}\right], \\ 0 & \text{if } t \in \left(\frac{1}{a}, 1\right]. \end{cases}
\]

Throughout this section we assume that the Banach space \( X \) satisfies the following properties:

(P1) \( X = \overline{W}^{\|\cdot\|} \);

(P2) there is a continuous decreasing function \( \alpha : [1,2] \to \mathbb{R} \) with \( \alpha(1) = 1 \) and \( \alpha(2) > 0 \) such that

\[
\alpha(a)\|f\| \leq \|f_a\| \leq \|f\|,
\]

for every \( f \in X \) and \( a \in [1,2] \). Then it is easy to check that \( f_a \in X \).

Now for any continuous function \( g \in X \) we set \( A_g = \{g_a : a \in [1,2]\} \). We need the following two lemmas, the proofs of which are straightforward.

Lemma 3.1. Let \( g \in X \) be continuous. Then the set \( A_g \) is compact.

Proof: Let \( g \in X \) be continuous. For any \( a \in [1,2] \), we have \( |g_a| \leq \|g\|_{\infty}\chi_{[0,1]} \), and then (1) implies

\[
\|g_a\| \leq \|g\|_{\infty}\|\chi_{[0,1]}\|.
\]

From the last inequality it follows that \( A_g \) has uniformly absolutely continuous norm. Let now \( \{g_{a_n}\} \) be a sequence of elements of \( A_g \). Choose a subsequence \( \{a_{n_k}\} \) of \( \{a_n\} \) which is convergent, say to \( a \). It is easy to check that \( g_{a_{n_k}} \to g_a \) almost everywhere in \([0,1]\), so that \( g_{a_{n_k}} \to g_a \) in measure. By Lemma 2.5, the thesis follows.

Lemma 3.2. Let \( g \in X \) be continuous and \( a_n \to a \) (\( a_n \in [1,2] \)). Then \( \|g_{a_n} - g_a\| \to 0 \).
**Proof:** Let \( g \in X \) be continuous and \( a_n \to a \ (a_n \in [1, 2]) \). Given \( \varepsilon > 0 \), as \( A_g \) has uniformly absolutely continuous norm, there exists \( \delta > 0 \) such that \( \|g_{a} \chi_{D}\| < \varepsilon \) and \( \|g_{a_n} \chi_{D}\| < \varepsilon \) for all \( n \in \mathbb{N} \) whenever \( D \in \Sigma \) and \( \mu(D) < \delta \). Find an index \( \nu \) such that for all \( n \geq \nu \) we have \( \frac{1}{a_n} \in (1/a - \delta/2, 1/a + \delta/2) \) and \( |g(a_n t) - g(at)| \leq \varepsilon \) for all \( t \in [0, 1] \) with \( t \leq 1/a - \delta/2 \). Then \( \sup_{[0,1/a-\delta/2]} |g(a_n(t)) - g(a(t))| \leq \varepsilon \) and so

\[
\| (g_{a_n} - g_a) \chi_{[0,1/a-\delta/2]} \| \leq \varepsilon \| \chi_{[0,1]} \|.
\]

Hence for every \( n \geq \nu \) we have

\[
\|g_{a_n} - g_a\| \leq \| (g_{a_n} - g_a) \chi_{[0,1/a-\delta/2]} \| + \| (g_{a_n} - g_a) \chi_{[1/a-\delta/2,1/a+\delta/2]} \| \\
\leq \varepsilon \| \chi_{[0,1]} \| + 2\varepsilon,
\]

and the thesis follows.

\[\square\]

**Remark 3.3.** If \( a_n \to a \ (a_n \in [1, 2]) \) by the same argument of Lemma 3.2 we have

\[
\| \chi_{(1/a_n,1]} - \chi_{(1/a,1]} \| \to 0.
\]

We now define a mapping \( Q : B(X) \to B(X) \) and establish the properties of \( Q \) we need. The explicit formula of a retraction \( R \), of which we can estimate the H-measure of noncompactness (that is, the infimum of all \( k \geq 1 \) for which \( R \) is a \( k \)-ball contractive retraction) and the lower H-measure of noncompactness, will depend on a suitable compact mapping \( P_X : B(X) \to X \) satisfying the hypotheses of the subsequent Theorem 3.6. To define \( Q : B(X) \to B(X) \) we set

\[
(Qf)(t) = f_2/(1+\|f\|)(t), \text{ for all } t \in [0,1].
\]

We clearly have \( Qf = f \) for all \( f \in S(X) \).

**Proposition 3.4.** The mapping \( Q \) is continuous.

**Proof:** Let \( \{f_n\} \) be a sequence of elements of \( B(X) \) such that \( \|f_n - f\| \to 0 \). Let \( \varepsilon > 0 \). By Lemma 2.6 there is a continuous \( g \in B(X) \) such that \( \|f - g\| \leq \varepsilon \). Choose and index \( \nu \) such that for all \( n \geq \nu \) we have \( \|f - f_n\| \leq \varepsilon \), by Lemma 3.2 we may also assume \( \|g_{2/(1+\|f_n\|)} - g_{2/(1+\|f\|)}\| \leq \varepsilon \). Using the last inequality and the right hand side of \( (2) \) we get, for all \( n \geq \nu \)

\[
\|Qf_n - Qf\| \leq \| (f_n)_{2/(1+\|f_n\|)} - f_{2/(1+\|f_n\|)} \| + \| f_{2/(1+\|f_n\|)} - g_{2/(1+\|f_n\|)} \| \\
+ \| g_{2/(1+\|f_n\|)} - g_{2/(1+\|f\|)} \| + \| g_{2/(1+\|f\|)} - f_{2/(1+\|f\|)} \| \\
= \| (f_n - f)_{2/(1+\|f_n\|)} \| + \| (f - g)_{2/(1+\|f\|)} \| \\
+ \| g_{2/(1+\|f_n\|)} - g_{2/(1+\|f\|)} \| + \| (g - f)_{2/(1+\|f\|)} \| \leq 4\varepsilon,
\]

which gives the thesis. \[\square\]
**Proposition 3.5.** Let \( A \subset B(X) \). Then
\[
\alpha(2) \gamma(A) \leq \gamma(QA) \leq \gamma(A).
\]

**Proof:** Let \( A \subset B(X) \). We prove the right inequality. Let \( \beta > \gamma(A) \). By Lemma 2.6, \( C([0,1]) \) is dense in \( X \), thus there exists a \( \beta \)-net \( \{\varphi_1, \ldots, \varphi_p\} \) for \( A \) in \( C([0,1]) \). By Lemma 3.1 the set \( \bigcup_{i=1}^{p} A_{\varphi_i} \) is compact, hence given \( \delta > 0 \) we can choose a \( \delta \)-net \( \{\psi_1, \ldots, \psi_q\} \) for \( \bigcup_{i=1}^{p} A_{\varphi_i} \) in \( X \). We now show that \( \{\psi_1, \ldots, \psi_q\} \) is a \( (\beta + \delta) \)-net for \( QA \) in \( X \).

Let \( g \in QA \) and let \( f \in A \) be such that \( Qf = g \). Fix \( i \in \{1, \ldots, p\} \) such that \( \|f - \varphi_i\| \leq \beta \). Since \( (\varphi_i)_{2/(1+\|f\|)} \in A_{\varphi_i} \) we can find \( j \in \{1, \ldots, q\} \) such that
\[
\|(\varphi_i)_{2/(1+\|f\|)} - \psi_j\| \leq \delta.
\]
Then
\[
\|Qf - \psi_j\| \leq \|(\varphi_i)_{2/(1+\|f\|)} - (\varphi_i)_{2/(1+\|f\|)}\| + \|(\varphi_i)_{2/(1+\|f\|)} - \psi_j\| \\
\leq \|f - \varphi_i\| + \delta \leq \beta + \delta.
\]
Therefore \( \gamma(QA) \leq \beta + \delta \), so \( \gamma(QA) \leq \gamma(A) \).

We now prove the left inequality. Let \( \eta > \gamma(QA) \). As \( C([0,1]) \) is dense in \( X \), there exists an \( \eta \)-net \( \{\lambda_1, \ldots, \lambda_n\} \) for \( QA \) in \( C([0,1]) \). For \( i = 1, \ldots, n \), set \( (\lambda_i)^b(t) = \lambda_i(bt) \) for \( t \in [0,1] \) and \( b \in [1/2, 1] \). Since each \( (\lambda_i)^b \) is a continuous mapping, the set \( \bigcup_{i=1}^{n} \{(\lambda_i)^b : b \in [1/2, 1]\} \) is compact with respect to the \( \| \cdot \|_\infty \) norm and hence is compact in \( X \). Hence for any \( \delta > 0 \) we can choose a \( \delta \)-net \( \{\xi_1, \ldots, \xi_m\} \) for \( \bigcup_{i=1}^{n} \{(\lambda_i)^b : b \in [1/2, 1]\} \) in \( X \).

We now show that \( \{\xi_1, \ldots, \xi_m\} \) is an \( (\eta/\alpha(2) + \delta) \)-net for \( A \) in \( X \).

Let \( f \in A \). Fix \( i \in \{1, \ldots, n\} \) such that \( \|Qf - \lambda_i\| \leq \eta \). Since
\[
(\lambda_i)^{(1+\|f\|)/2} \in \{(\lambda_i)^b : b \in [1/2, 1]\}
\]
we can find \( j \in \{1, \ldots, m\} \) such that \( \|(\lambda_i)^{(1+\|f\|)/2} - \xi_j\| \leq \delta \). Then
\[
\|f - \xi_j\| \leq \|f - (\lambda_i)^{(1+\|f\|)/2}\| + \|(\lambda_i)^{(1+\|f\|)/2} - \xi_j\| \\
\leq \frac{1}{\alpha(2)} \|f_{2/(1+\|f\|)} - (\lambda_i)^{(1+\|f\|)/2}_{2/(1+\|f\|)}\| + \delta \\
\leq \frac{1}{\alpha(2)} \|Qf - \lambda_i\| + \delta \leq \frac{\eta}{\alpha(2)} + \delta.
\]
Therefore \( \gamma(A) \leq \eta/\alpha(2) + \delta \), so \( \alpha(2) \gamma(A) \leq \gamma(QA) \).

**Theorem 3.6.** Let \( P_X : B(X) \to X \) be a compact mapping with \( P_X f = 0 \) for all \( f \in S(X) \), and
\[
\|Qf + P_X f\| \geq m,
\]
for some \( m \in (0, 1] \) and all \( f \in B(X) \). Then the mapping \( R : B(X) \to S(X) \) defined by

\[
Rf = \frac{Qf + P_X f}{\|Qf + P_X f\|},
\]

is a \((1/m)\) -ball contractive retraction. Moreover \( \omega(R) \geq \alpha(2)/l \) whenever \( \|Qf + P_X f\| \leq l \) for all \( f \in B(X) \). In particular, if \( \|Qf + P_X f\| = 1 \) for all \( f \in B(X) \), the retraction \( R \) is 1-ball contractive and \( \omega(R) \geq \alpha(2) \).

**Proof:** Clearly the mapping \( R \) defined in (5) is a retraction. Let \( A \subseteq B(X) \). Since \( P_X \) is compact, it follows from Proposition 3.5 that

\[
\alpha(2) \gamma(A) \leq \gamma((Q + P_X) A) \leq \gamma(A).
\]

Moreover by the definition of \( R \) and by (4) we get

\[
RA \subset [0, \frac{1}{m}] \cdot (Q + P_X) A.
\]

Using the properties of \( \gamma \), from (6) it follows \( \gamma(RA) \leq (1/m) \gamma(A) \). Similarly if \( \|Qf + P_X f\| \leq l \) for all \( f \in B(X) \) we have

\[
(Q + P_X) A \subset [0, l] \cdot RA.
\]

Therefore \( (\alpha(2)/l) \gamma(A) \leq \gamma(RA) \), and the proof is complete.

Observe that \( \|Qf + P_X f\| = 1 \) for \( f \in S(X) \), so in condition (4) we necessarily have \( m \leq 1 \).

**Remark 3.7.** Whenever in a Banach space \( X \) we find \( \alpha(a) \|f\| = \|f_a\| \), for all \( f \in B(X) \) (a stronger condition than (2)) we modify the mapping \( Q \) defined in (3) by setting

\[
(Qf)(t) = \frac{1}{\alpha(2/(1 + \|f\|))} f_{2/(1 + \|f\|)}(t), \quad \text{for all } t \in [0, 1].
\]

As no confusion can arise we keep denoting this mapping by \( Q \). Then \( \|Qf\| = \|f\| \) for all \( f \in B(X) \). Clearly \( Q \) is still a continuous mapping and, by slight modifications of the previous arguments and of Proposition 3.5, we get \( \gamma(QA) = \gamma(A) \). This allow us to obtain a better estimate of the lower \( H \)-measure of noncompactness of the retraction \( R \) defined as in (5). In fact, under the same hypotheses of Theorem 3.6, we get \( \omega(R) \geq 1/l \).

**Corollary 3.8.** The retraction \( R \) defined in (5) is a proper mapping.

4. **The Orlicz spaces \( L_\Phi \).**

Let \( \Phi : [0, \infty) \to [0, \infty) \) be a continuous strictly increasing Young’s function. Assume that \( \Phi \) satisfies the \( \Delta_2 \)-condition, that is, there is \( c \in [0, \infty) \) such that \( \Phi(2x) \leq c \Phi(x) \) \((x \geq 0)\). For \( f \in M_0 \) set

\[
M^\Phi(f) = \int_{[0,1]} \Phi\left(\left|f(t)\right|\right) dt.
\]
Then

\[ \rho_\Phi(f) = \inf \left\{ u > 0 : M^\Phi \left( \frac{f}{u} \right) \leq 1 \right\} \ (f \in \mathcal{M}_0^+) \]

is a Banach function norm, and the Banach function space

\[ L_\Phi := L_\Phi[0,1] = \left\{ f \in \mathcal{M}_0 : \rho_\Phi(|f|) < \infty \right\} \]

is the Orlicz space generated by \( \Phi \) endowed with the Luxemburg norm \( \|f\|_\Phi = \rho_\Phi(|f|) \).

The Orlicz space \( L_\Phi \) is of absolutely continuous norm (see for example [14]). Then by Lemma 2.4 the space \( L_\Phi \) satisfies property (P2). The following lemma proved in [12] shows that (P2) holds in \( L_\Phi \).

**Lemma 4.1.** ([12, Lemma 2.3]) Let \( f \in L_\Phi \) and \( a \in [1,2] \). Then

\[ \frac{1}{a} \|f\|_\Phi \leq \|f_a\|_\Phi \leq \|f\|_\Phi. \]

Let \( Q : B(L_\Phi) \to B(L_\Phi) \) be defined as in (3) and define \( P_\Phi : B(L_\Phi) \to L_\Phi \) by

\[ P_\Phi f = \begin{cases} 
\Phi^{-1} \left( \frac{2}{1 - \|f\|_\Phi} \left( 1 - M^\Phi(Qf) \right) \right) \chi_{\left( (1 + \|f\|_\Phi)/2, 1 \right]} & \text{if } f \in B(L_\Phi) \setminus S(L_\Phi) \\
0 & \text{if } f \in S(L_\Phi). 
\end{cases} \]

**Lemma 4.2.** The mapping \( P_\Phi \) is compact.

**Proof:** We prove that \( P_\Phi B(L_\Phi) \) is relatively compact and \( P_\Phi \) is continuous. Let \( \{g_n\} \) be a sequence of elements of \( P_\Phi B(L_\Phi) \) and \( \{f_n\} \) be a sequence of elements of \( B(L_\Phi) \) such that \( P_\Phi f_n = g_n \), for all \( n \). Since \( 0 \leq \|f_n\|_\Phi \leq 1 \) and \( 0 \leq M^\Phi(Qf_n) \leq \|Qf_n\|_\Phi \leq 1 \) for all \( n \), we can choose subsequences \( \{|f_{n_k}\|_\Phi\} \), \( \{|Qf_{n_k}\|_\Phi\} \) and \( \{M^\Phi(Qf_{n_k})\} \) which converge, say to \( b, c \) and \( c_\Phi \), respectively.

If \( b = 1 \) then by Lemma 4.1, \( \|Qf_{n_k}\|_\Phi \to 1 \) and consequently

\[ M^\Phi(Qf_{n_k}) \to 1 \]

Since \( M^\Phi(P_\Phi f_{n_k}) = 1 - M^\Phi(Qf_{n_k}) \) we have \( M^\Phi(P_\Phi f_{n_k}) \to 0 \) and hence \( \|P_\Phi f_{n_k}\|_\Phi \to 0 \). This implies that \( \{g_{n_k}\} \) converges in norm to the null function. Assume \( b < 1 \) and write

\[
\left\| P_\Phi f_{n_k} - \Phi^{-1} \left( \frac{2}{1 - b} (1 - c_\Phi) \right) \chi_{\left( (1 + b)/2, 1 \right]} \right\|_\Phi
\]

\[
= \left\| \Phi^{-1} \left( \frac{2}{1 - \|f_{n_k}\|_\Phi} \left( 1 - M^\Phi(Qf_{n_k}) \right) \right) \chi_{\left( (1 + \|f_{n_k}\|_\Phi)/2, 1 \right]} \right\|_\Phi
\]

\[
- \Phi^{-1} \left( \frac{2}{1 - b} (1 - c_\Phi) \right) \chi_{\left( (1 + b)/2, 1 \right]} \right\|_\Phi.
\]

By Remark 3.3 we have

\[
\left\| \chi_{\left( (1 + \|f_{n_k}\|_\Phi)/2, 1 \right]} - \chi_{\left( (1 + b)/2, 1 \right]} \right\|_\Phi \to 0
\]
and by the continuity of $\Phi^{-1}$ we also have
\[
\Phi^{-1}\left(\frac{2}{1 - \|f_{n_k}\|_\Phi} (1 - M^{\Phi}(Qf_{n_k}))\right) \to \Phi^{-1}\left(\frac{2}{1 - b} (1 - c_\Phi)\right).
\]
Thus we get
\[
\left\| P_{\Phi} f_{n_k} - \Phi^{-1}\left(\frac{2}{1 - b} (1 - c_\Phi)\right) \chi((1+b/2), 1]\right\|_\Phi \to 0.
\]
We have proved that $P_{\Phi} B(L_{\Phi})$ is relatively compact.

Let now $\{f_n\}$ be a sequence of elements of $B(L_{\Phi})$ such that $\|f_n - f\|_\Phi \to 0$, then, as the $\Delta_2$-condition holds, $M^{\Phi}(f_n) \to M^{\Phi}(f)$. An argument similar to that of the first part of the proof implies $\|P_{\Phi} f_n - P_{\Phi} f\|_\Phi \to 0$. The proof is complete. $lacksquare$

**Lemma 4.3.** Let $f \in B(L_{\Phi})$, then
\[
\|Qf + P_{\Phi} f\|_\Phi = 1.
\]

**Proof:** Observe that, for any $u > 0$ we have
\[
M^{\Phi}\left(\frac{Qf + P_{\Phi} f}{u}\right) = M^{\Phi}\left(\frac{Qf}{u}\right) + M^{\Phi}\left(\frac{P_{\Phi} f}{u}\right).
\]
Now for $u = 1$ we get
\[
M^{\Phi}(Qf + P_{\Phi} f) = \int_{\left[(1 + \|f\|_\Phi)/2, 1\right]} \Phi\left(\Phi^{-1}\left(\frac{2}{1 - \|f\|_\Phi} (1 - M^{\Phi}(Qf))\right)\right) dt + M^{\Phi}(Qf)
\]
\[
= \int_{\left[(1 + \|f\|_\Phi)/2, 1\right]} \frac{2}{1 - \|f\|_\Phi} (1 - M^{\Phi}(Qf)) dt + M^{\Phi}(Qf) = 1
\]
It follows that $\|Qf + P_{\Phi} f\|_\Phi \leq 1$. On the other hand if $0 < u < 1$
\[
M^{\Phi}\left(\frac{Qf + P_{\Phi} f}{u}\right) > M^{\Phi}(Qf + P_{\Phi} f),
\]
consequently $\|Qf + P_{\Phi} f\|_\Phi = 1$. $lacksquare$

From Lemmas 4.1, 4.2 and 4.3 and Theorem 3.6 we obtain the following.

**Theorem 4.4.** The mapping $R : B(L_{\Phi}) \to S(L_{\Phi})$ defined by
\[
Rf = Qf + P_{\Phi} f
\]
is a 1-ball contractive retraction and $\omega(R) \geq 1/2$.

Observe that, if $\Phi(t) = t^p$ where $1 \leq p < \infty$, then $L_{\Phi}$ is the Lebesgue space $L_p := L_p[0, 1]$, with the standard norm $\| \cdot \|_p$. But in this case an easy computation shows that $(1/a)^{1/p}\|f\|_p = \|f_a\|_p$. Hence, according to Remark 3.7, a stronger result on the characteristic $\omega(R)$ holds. Define $Q : B(L_p) \to B(L_p)$ (as in (7)) by
\[
(Qf)(t) = \left(\frac{2}{1 + \|f\|_p}\right)^{1/p} f_{2/(1+\|f\|_p)}(t), \quad \text{for all } t \in [0, 1].
\]
Next define $P_p : B(L_p) \to L_p$ by

$$P_p f = \begin{cases} 
\frac{2}{1 - \|f\|_p} \left(1 - \|f\|_p^p\right)^{1/p} X_{(1+\|f\|_p)/2,1} & \text{if } f \in B(L_p) \setminus S(L_p) \\
0 & \text{if } f \in S(L_p).
\end{cases}$$

Then the following theorem holds.

**Theorem 4.5.** The mapping $R : B(L_p) \to S(L_p)$ $(1 \leq p < \infty)$ defined by

$$Rf = Qf + P_p f$$

is a 1-ball contractive retraction and $\omega(R) = 1$.

The results obtained in the Lebesgue spaces $L_p$ can be generalised to the weighted spaces. Let $\rho$ be a measurable weighting function. We consider the weighted Lebesgue space

$$L_p(\rho) := L_p([0, 1], \rho) \ (1 \leq p < \infty)$$

which consists of all $f \in M_0$ such that $\rho^{1/p} f \in L_p$, endowed with the norm

$$\|f\|_{L_p(\rho)} = \left(\int_{[0,1]} \rho(t) |f(t)|^p \, dt\right)^{1/p}.$$ 

The space $L_p(\rho)$ has absolutely continuous norm.

We define a mapping $Q_\rho : B(L_p(\rho)) \to B(L_p(\rho))$ by a slight modification of (7)

$$(Q_\rho f)(t) = \left(\rho^{2/(1+\|f\|_{L_p(\rho)})}(t)/\rho(t)\right)^{1/p} \left(\frac{2}{1 + \|f\|_{L_p(\rho)}}\right)^{1/p} f_{2/(1+\|f\|_{L_p(\rho)})}(t) \text{ for all } t \in [0, 1]$$

and define $P_\rho : B(L_p(\rho)) \to L_p(\rho)$ by

$$P_\rho f = \begin{cases} 
\left(\frac{2}{1 - \|f\|_{L_p(\rho)}^p}\right)^{1/p} \left(1 - \frac{\|f\|_{L_p(\rho)}^p}{\rho(t)}\right)^{1/p} X_{(1+\|f\|_p)/2,1} & \text{if } f \in B(L_p(\rho)) \setminus S(L_p(\rho)) \\
0 & \text{if } f \in S(L_p(\rho)).
\end{cases}$$

Set

$$C([0, 1], \rho) = \{g/\rho^{1/p} : g \in C[0, 1]\}$$

and

$$W(\rho) = \{s/\rho^{1/p} : s \in W\}.$$ 

Then $C([0, 1], \rho)$ is dense in $L_p([0, 1], \rho)$ and $L_p([0, 1], \rho) = \overline{W(\rho)^{1/p} L_p(\rho)}$. Moreover for a continuous function $g$, the set $A_g(\rho) = \{g_a/\rho^{1/p} : a \in [1, 2]\}$ is compact. Then the same arguments of Section 3 allow us to obtain the following.
**Corollary 4.6.** The mapping

\[ R : B(L_p(\rho)) \to S(L_p(\rho)) \quad (1 \leq p < \infty) \]

defined by \( Rf = Q_\rho f + P_\rho f \) is a 1-ball contractive retraction with \( \omega(R) = 1 \).

In this section we have improved the results in the \( L_p \) and \( L_\Phi \) spaces of \([17, 12]\), respectively. Though the mapping \( Q \) is the same as the one introduced in those papers, here we construct in both cases a different retraction \( R \) and, above all, our proofs are based on different ideas and techniques.

5. The Lorentz spaces \( L^{p,q} \).

Let \( f^* \) denote the decreasing rearrangement of a function \( f \in \mathcal{M}_0 \), given by

\[
 f^* (t) = \inf \left\{ s \geq 0 : \mu \{ |f(x)| > s \} \leq t \right\}
\]

The Lorentz space \( L^{p,q} := L^{p,q}([0,1]) \) (\( 1 \leq q \leq p < \infty \)) consists of all \( f \in \mathcal{M}_0 \) for which the quantity

\[
 \|f\|_{p,q} = \left( \frac{q}{p} \int_{[0,1]} \left( t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q}
\]

is finite. As the Lorentz space \( L^{p,q} \) is reflexive (see for example \([14]\)) from \([3, Corollary 4.4]\) it follows that it has absolutely continuous norm. Hence by Lemma 2.4 the space \( L^{p,q} \) satisfies property (P1).

**Lemma 5.1.** Let \( f \in L^{p,q} \) and \( a \in [1,2] \), then

\[
 \left( \frac{1}{a} \right)^{1/p} \|f\|_{p,q} = \|f_a\|_{p,q}.
\]

**Proof:** Let \( f \in L^{p,q} \). We observe that we have \((f_a)^* = (f^*)_a\). Then the lemma follows by a direct computation of \( \|f_a\|_{p,q}^q \). Indeed we have

\[
 \|f_a\|_{p,q}^q = \frac{q}{p} \int_{[0,1]} t^{(q/p)-1} ((f_a)^*(t))^q \, dt = \frac{q}{p} \int_{[0,1]} t^{(q/p)-1} ((f^*)_a(t))^q \, dt
\]

\[
 = \frac{q}{p} \int_{[0,1/a]} t^{(q/p)-1} (f^*(at))^q \, dt
\]

\[
 = \left( \frac{1}{a} \right)^{q/p} \frac{q}{p} \int_{[0,1]} t^{(q/p)-1} (f^*(t))^q \, dt = \left( \frac{1}{a} \right)^{(q/p)} \|f\|_{p,q}^q,
\]

hence the thesis.

In view of Lemma 5.1 and Remark 3.7 we define \( Q : B(L^{p,q}) \to B(L^{p,q}) \) (as in (7)) by

\[
 (Qf)(t) = \left( \frac{2}{1 + \|f\|_{p,q}} \right)^{1/p} f_{2/(1+\|f\|_{p,q})}(t), \quad \text{for all} \ t \in [0,1].
\]
Next define \( P_{p,q} : B(L^{p,q}) \to L^{p,q} \)

\[
P_{p,q} f = \begin{cases} 
\left( \frac{2}{1 - \|f\|_{p,q}} \right)^{1/p} (1 - \|f\|_{p,q}^{q/1/q})^{1/q} \chi \left( (1 + \|f\|_{p,q})/2, 1 \right) & \text{if } f \in B(L^{p,q}) \setminus S(L^{p,q}) \\
0 & \text{if } f \in S(L^{p,q}).
\end{cases}
\]

We have that the mapping \( P_{p,q} \) is compact and \( \|Qf + P_{p,q} f\|_{p,q} = 1 \) for all \( f \in B(L^{p,q}) \). Hence by Theorem 3.6 and Remark 3.7 we obtain the following.

**Theorem 5.2.** The mapping

\[
R : B(L^{p,q}) \to S(L^{p,q}) \quad (1 \leq q \leq p < \infty)
\]

defined by

\[
Rf = Qf + P_{p,q} f
\]

is a 1-ball contractive retraction and \( \omega(R) = 1 \).

The questions whether or not \( W(X) = 1 \) in any infinite-dimensional Banach space \( X \), and eventually if this value is always a minimum remain open.

We conclude this section with some remarks on fixed point free self-mappings of the unit ball \( B(X) \). In [1, Theorem 3] the following theorem has been proved.

**Theorem 5.3.** Let \( X \) be an infinite-dimensional Banach space and \( \varepsilon > 0 \). Then there exists a fixed point free 1-ball contraction \( F : B(X) \to B(X) \) with \( \omega(F) \geq 1 - \varepsilon \).

We have that, in some Banach spaces, the best value \( \omega(F) = 1 \) can be attained by a fixed point free 1-ball contraction \( F : B(X) \to B(X) \). Indeed if \( R : B(X) \to S(X) \) is a \( k \)-ball contractive retraction, then \( F = -R : B(X) \to B(X) \) is a fixed point free \( k \)-ball contraction. As a consequence of Corollary 4.6 and Theorem 5.2 we obtain the following.

**Corollary 5.4.** Let \( X \) denote either the weighted Lebesgue space \( L_p(\rho) \) \((1 \leq p < \infty)\) or the Lorentz space \( L^{p,q} \) \((1 \leq q \leq p < \infty)\). Then there exists a fixed point free 1-ball contraction \( F : B(X) \to B(X) \) with \( \omega(F) = 1 \).

6. **Banach spaces with \((1 + \varepsilon)\)-ball contractive retractions.**

In this section we consider \( X \) to be the space of all functions of absolutely continuous norm of a Banach function space \( Y \), where \( Y \) is either the grand \( L^p \) space or the Marcinkiewicz spaces \( M_\beta \). Applying Theorem 3.6 we prove that, in both cases, for any \( \varepsilon > 0 \) there is a \((1 + \varepsilon)\)-ball contractive retraction \( R \) with positive \( H \)-lower measure of noncompactness.

Let \( 1 < p < \infty \). The grand \( L^p \) space, which will be denoted by \( L^p) := L^p([0, 1]) \), introduced in [11], is defined as the space of all functions \( f \in \mathcal{M}_0 \) such that

\[
\|f\|_p = \sup_{0<\varepsilon<p-1} \left( \varepsilon \int_{[0,1]} |f(t)|^{p-\varepsilon} dt \right)^{1/(p-\varepsilon)} < \infty.
\]
We denote by \( X^p \) the set of all functions in \( L^p \) of absolutely continuous norm and by \( W \) the subset of \( L^p \) of all simple functions.

**Lemma 6.1.** The subspace \( X^p \) coincides with \( \overline{W}^{\|\cdot\|_p} \), and the inclusion \( X^p \subset L^p \) is proper.

**Proof:** Let \( \sigma > 0 \) and set \( \delta = (\sigma/(p-1))^p \). Let \( D \in \Sigma \) with \( \mu(D) < \delta \). As
\[
\sup_{0 < \varepsilon < p-1} \varepsilon^{1/(p-\varepsilon)} = p-1 \quad \text{and} \quad \sup_{0 < \varepsilon < p-1} \mu(D)^{1/(p-\varepsilon)} = \mu(D)^{1/p}
\]
we have
\[
\|X_D\|_p = \sup_{0 < \varepsilon < p-1} (\varepsilon \mu(D))^{1/(p-\varepsilon)} \leq (p-1)\mu(D)^{1/p} < \sigma.
\]
This shows that \( \chi_{[0,1]} \) has absolutely continuous norm, hence by Lemma 2.4 (iii) it follows \( X^p = \overline{W}^{\|\cdot\|_p} \). To end the proof it suffices to note that the function \( t^{-1/p} \in L^p \) has not absolutely continuous norm.

**Lemma 6.2.** Let \( f \in X^p \) and \( a \in [1, 2] \),
\[
\frac{1}{a} \|f\|_p \leq \|fa\|_p \leq \|f\|_p.
\]

**Proof:** For any \( f \in X^p \) and \( a \in [1, 2] \) we have
\[
\|fa\|_p = \sup_{0 < \varepsilon < p-1} (\varepsilon \int_{[0, \frac{a}{\varepsilon}]} |f(at)|^{p-\varepsilon} \, dt) \leq \varepsilon \mu(D)^{1/(p-\varepsilon)} \leq \|f\|_p.
\]
On the other hand we find
\[
\|f\|_p = \sup_{0 < \varepsilon < p-1} a^{1/(p-\varepsilon)} (\varepsilon \int_{[0,1]} |f(at)|^{p-\varepsilon} \, dt) \leq a\|fa\|_p,
\]
which completes the proof.

Let \( Q : B(X^p) \rightarrow B(X^p) \) be defined as in (3) and define for every \( 0 < u < \infty \) the mapping \( (P_p)_u : B(X^p) \rightarrow X^p \) by
\[
(P_p)_uf = \begin{cases} \frac{u^{1-\|Qf\|_p}}{\|\chi\{(1+\|f\|_p)/2,1\}\|_p} X^{(1+\|f\|_p)/2,1} & \text{if } f \in B(X^p) \setminus S(X^p) \\ 0 & \text{if } f \in S(X^p). \end{cases}
\]

**Lemma 6.3.** For any \( 0 < u < \infty \), the mapping \( (P_p)_u \) is compact, and for \( f \in B(X^p) \)
\[
\| (P_p)_uf\|_p = u(1-\|Qf\|_p)
\]

**Proof:** The proof that \( (P_p)_u \) is compact is similar to the proof of Lemma 4.2. A direct calculation gives the norm of \( (P_p)_uf \).
**Lemma 6.4.** Let $0 < u < \infty$. For any $f \in B(X^p)$

$$\max\{1, u\} \geq \|Qf + (P_p)_u f\|_p \geq \frac{u}{u + 1}$$

**Proof:** Let $f \in B(X^p)$, then

$$\|Qf + (P_p)_u f\|_p = \sup_{0 < \varepsilon < p - 1} \left( \varepsilon \int_{[0,1]} |(Qf)(t)|^{p-\varepsilon} dt + \varepsilon \int_{[0,1]} |(P_p)_u f(t)|^{p-\varepsilon} dt \right)^{1/(p-\varepsilon)}.$$

Now for any fixed $0 < \varepsilon < p - 1$

$$\varepsilon \int_{[0,1]} |(Qf)(t)|^{p-\varepsilon} dt \leq \varepsilon \int_{[0,1]} |(Qf)(t)|^{p-\varepsilon} dt + \varepsilon \int_{[0,1]} |(P_p)_u f(t)|^{p-\varepsilon} dt$$

and passing to the $1/(p - \varepsilon)$-power we have

$$\left( \varepsilon \int_{[0,1]} |(Qf)(t)|^{p-\varepsilon} dt \right)^{1/(p-\varepsilon)} \leq \left( \varepsilon \int_{[0,1]} |(Qf)(t)|^{p-\varepsilon} dt + \varepsilon \int_{[0,1]} |(P_p)_u f(t)|^{p-\varepsilon} dt \right)^{1/(p-\varepsilon)}.$$

Taking the supremum over $\varepsilon$ we get $\|Qf + (P_p)_u f\|_p \geq \|Qf\|_p$. Analogously we get

$$\|Qf + (P_p)_u f\|_p \geq \|Qf\|_p,$$

Then

$$\|Qf + (P_p)_u f\|_p \geq \max\left\{\|Qf\|_p, u(1 - \|Qf\|_p)\right\} \geq u/u + 1.$$ 

On the other hand it easily follows

$$\|Qf + (P_p)_u f\|_p \leq \|Qf\|_p + u(1 - \|Qf\|_p) \leq \max\{1, u\}.$$

By Lemmas 6.1 and 6.2, the Banach space $X^p$ satisfies properties (P1) and (P2). Hence by Lemmas 6.3 and 6.4 and Theorem 3.6 we have that the mapping $R_u : B(X^p) \to S(X^p)$ defined by

$$R_u f = \frac{Qf + (P_p)_u f}{\|Qf + (P_p)_u f\|_p}$$

is $(u + 1)/u-$ball contractive with $\omega(R_u) \geq \min\{1/2, 1/(2u)\}$. As $\lim_{u \to \infty} (u + 1)/u = 1$ we obtain the following theorem.

**Theorem 6.5.** For any $\varepsilon > 0$ there is a retraction

$$R : B(X^p) \to S(X^p) \ (1 < p < \infty)$$

which is $(1 + \varepsilon)$-ball contractive with $\omega(R) > 0$. 
Remark 6.6. The same result of Theorem 6.5 can be proved in the small Lebesgue space $L^p$ ($1 < p < \infty$) introduced in [7], in which the norm is defined as

$$\|f\|_{p'} = \sup_{g \in L^p} \int_{[0,1]} \frac{f(t)g(t)\, dt}{\|g\|_p}.$$  

We recall that the spaces $L^p$ have absolutely continuous norm, and the spaces $L^p'$ are characterised as dual spaces of $L^p$ (see [5]).

An analogous result holds in the Marcinkiewicz space

$$M_\beta := M_\beta([0, 1]) (0 < \beta < 1)$$

which consists of all $f \in M_0$ for which

$$\|f\|_\beta = \sup \frac{1}{\mu(E)^\beta} \int_E |f(t)| \, dt < \infty.$$  

where the supremum is taken over all $E \in \Sigma$ with $\mu(E) > 0$. We denote by $X_\beta$ the set of all functions in $M_\beta$ of absolutely continuous norm and $W$ the subset of $M_\beta$ of all simple functions.

Lemma 6.7. The subspace $X_\beta$ coincides with $W_{\|\cdot\|_\beta}$, and the inclusion $X_\beta \subset M_\beta$ is proper.

Proof: We prove that for every $D \in \Sigma$

$$\|\chi_D\|_\beta = \mu(D)^{1-\beta}. \tag{8}$$

By definition we have

$$\|\chi_D\|_\beta = \sup \frac{1}{\mu(E)^\beta} \mu(D \cap E).$$

Choose for every $n \in N$ a set $E_n \in \Sigma$ such that

$$\|\chi_D\|_\beta - \frac{1}{n} \leq \frac{1}{\mu(E_n)^\beta} \mu(D \cap E_n) \leq \|\chi_D\|_\beta.$$

Set $D_n = D \cap E_n$. As $D_n \subset E_n$ we get $1/(\mu(E_n)^\beta) \leq 1/(\mu(D_n)^\beta)$. Consequently,

$$\|\chi_D\|_\beta - \frac{1}{n} \leq \frac{1}{\mu(E_n)^\beta} \mu(D_n) \leq \frac{1}{\mu(D_n)^\beta} \mu(D_n) \leq \|\chi_D\|_\beta.$$

As $n$ goes to infinity we get (8). From (8) it obviously follows that $\chi_{[0,1]}$ has absolutely continuous norm, hence (iii) of Lemma 2.4 gives $X_\beta = W_{\|\cdot\|_\beta}$. As pointed out in [2] the space $M_\beta$ has not absolutely continuous norm.

It easy to check that the following lemma holds.
Lemma 6.8. Let \( f \in X_\beta \) and \( a \in [1, 2] \),
\[
\left( \frac{1}{a} \right)^{1-\beta} \|f\|_\beta \leq \|f_a\|_\beta \leq \|f\|_\beta.
\]

Now let \( Q : B(X_\beta) \to B(X_\beta) \) be defined as in (3) and define for every \( 0 < u < \infty \)
the mapping \( (P_\beta)_u : B(X_\beta) \to X_\beta \) by
\[
(P_\beta)_uf = \begin{cases} 
  u \left( \frac{2}{1 - \|f\|_\beta} \right)^{1-\beta} (1 - \|Qf\|_\beta) \chi_{(1+\|f\|_\beta)/2,1]} & \text{if } f \in B(X_\beta) \setminus S(X_\beta) \\
  0 & \text{if } f \in S(X_\beta).
\end{cases}
\]

For every \( 0 < u < \infty \), the mapping \( (P_\beta)_u \) is compact and
\[
\|(P_\beta)_uf\|_\beta = u(1 - \|Qf\|_\beta).
\]

Moreover the following estimates of \( \|Qf + (P_\beta)_uf\|_\beta \) can be derived by an argument
similar to that of Lemma 6.4.

Lemma 6.9. Let \( 0 < u < \infty \). For any \( f \in B(X_\beta) \)
\[
\max\{1, u\} \geq \|Qf + (P_\beta)_uf\|_\beta \geq \frac{u}{u+1}.
\]

By Lemmas 6.7 and 6.8, the Banach space \( X_\beta \) satisfies properties (P1) and (P2). Then by the previous Lemma and Theorem 3.6 we have that the mapping \( R_u : B(X_\beta) \to S(X_\beta) \) defined by
\[
R_u f = \frac{Qf + (P_\beta)_uf}{\|Qf + (P_\beta)_uf\|_\beta}
\]
is \((u+1)/u\)-ball contractive with
\[
\omega(R_u) \geq \min\{1/(2^{1-\beta}u), 1/(2^{1-\beta})\}.
\]
As \( \lim_{u \to \infty} (u+1)/u = 1 \) we obtain the following.

Theorem 6.10. For any \( \varepsilon > 0 \) there is a retraction \( R : B(X_\beta) \to S(X_\beta) \) which
is \((1+\varepsilon)\)-ball contractive with \( \omega(R) > 0 \).

References


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