INEQUALITIES FOR THE BETA FUNCTION
OF \( n \) VARIABLES

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Abstract

We present various inequalities for Euler’s beta function of \( n \) variables. One of our theorems
states that the inequalities

\[
a_n \leq \frac{1}{\prod_{i=1}^{n} x_i} - B(x_1, \ldots, x_n) \leq b_n
\]

hold for all \( x_i \geq 1 \) (\( i = 1, \ldots, n; \ n \geq 3 \)) with the best possible constants \( a_n = 0 \) and
\( b_n = 1 - 1/(n - 1)! \). This extends a recently published result of Dragomir et al., who
investigated (*) for the special case \( n = 2 \).

1. Introduction

The classical beta function, which is also known as Euler’s integral of the first kind,
is defined for positive real numbers \( x \) and \( y \) by

\[
B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt.
\]

The beta function plays a central role in the theory of special functions and also has
applications in other fields, such as mathematical physics and probability theory; see
[4, 5, 8]. An extension of (1.1) to \( n \) variables is given by

\[
B(x_1, \ldots, x_n) = \int_{\Delta_{n-1}} \left( \prod_{i=1}^{n-1} t_i^{x_i-1} \right) \left( 1 - \sum_{i=1}^{n-1} t_i \right)^{x_n-1} \, dt_1 \cdots dt_{n-1}
\]

\((x_i > 0; \ i = 1, \ldots, n; \ n \geq 2)\), where

\[
\Delta_{n-1} = \left\{ (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} \mid t_1 \geq 0, \ldots, t_{n-1} \geq 0, t_1 + \cdots + t_{n-1} \leq 1 \right\}
\]

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denotes the standard simplex in $\mathbb{R}^{n-1}$. There exists a close connection between $B(x_1, \ldots, x_n)$ and the gamma function,

$$
\Gamma(x) = \int_0^\infty e^{-t x^{-1}} \, dt \quad (x > 0),
$$

as the elegant identity

$$
B(x_1, \ldots, x_n) = \frac{\Gamma(x_1) \cdots \Gamma(x_n)}{\Gamma(x_1 + \cdots + x_n)}
$$

reveals. A collection of the most important properties of the beta function of two and more variables is given, for instance, in [4, 8].

Various inequalities for $B(x, y)$ and $B_p(x, y) = \int_0^p t x^{-1}(1 - t)^{y-1} \, dt$ appear in the literature (see [12, 13, 15, 16, 19]), whereas inequalities for the beta function of three or more variables are difficult to find. The following interesting inequality for $B(x, y)$ was published in 2000 by Dragomir et al. [9]:

$$
0 \leq 1/(xy) - B(x, y) \leq 1/4 \quad (x, y \geq 1). \quad (1.2)
$$

The lower bound 0 is sharp, but the upper bound 1/4 can be improved. In [3] it is shown that the second inequality of (1.2) is valid with the best possible constant 0.08731. . . . It is natural to look for an extension of (1.2) to more than two variables. In this paper we determine the best possible constants $a_n$ and $b_n$ such that the double-inequality (*) holds for all $x_i \geq 1$ ($i = 1, \ldots, n; n \geq 3$). Furthermore, we establish several new inequalities for $B(x_1, \ldots, x_n)$, which are valid for all $n \geq 2$. In Section 3 we provide sharp constants $c_n$ in

$$
\alpha_n(c) = \frac{\prod_{i=1}^n x_i^{-1/2+x_i}}{\left(\sum_{i=1}^n x_i\right)^{-1/2+\sum_{i=1}^n x_i}} \leq B(x_1, \ldots, x_n) \leq \beta_n(c) = \frac{\prod_{i=1}^n x_i^{-1/2+x_i}}{\left(\sum_{i=1}^n x_i\right)^{-1/2+\sum_{i=1}^n x_i}},
$$

where $x_i \geq c > 0$ ($i = 1, \ldots, n$). Moreover, we determine the best possible upper and lower bounds for the ratio $B(\mu x_1, \mu x_2, \ldots, \mu x_n) / B(\nu x_1, \nu x_2, \ldots, \nu x_n)$, depending only on $\mu, \nu$ and $n$, and we establish that the inequalities

$$
B((x_1 + x_2)/2, \ldots, (x_n + y_n)/2) \leq \sqrt{B(x_1, \ldots, x_n) B(y_1, \ldots, y_n)}
$$

and

$$
B(x_1 + y_1, \ldots, x_n + y_n) \leq \frac{1}{2^n} (B(x_1, \ldots, x_n) + B(y_1, \ldots, y_n))
$$

are valid for all $x_i > 0$ ($i = 1, \ldots, n$). In order to prove our results we need some lemmas, which we present in the next section.
2. Lemmas

First, we collect a few basic properties of the gamma function and its logarithmic derivative \( \psi = \Gamma'/\Gamma \), which is known as the psi or digamma function.

**Lemma 2.1.** Let \( a > 0 \), \( b \geq 0 \) and \( x > 0 \) be real numbers and let \( n \geq 1 \) be an integer. Then we have

\[
\Gamma(ax + b) \sim \sqrt{2\pi e^{-ax}}(ax)^{ax+b-1/2} \quad (x \to \infty),
\]

\[
\log \Gamma(x) \sim (x - \frac{1}{2}) \log x - x + \frac{1}{2} \log (2\pi) + \frac{1}{12x} + \cdots \quad (x \to \infty),
\]

\[
\Gamma(2x) = \frac{1}{2\sqrt{\pi}} 4^n \Gamma(x) \Gamma(x + 1/2),
\]

\[
\lim_{x \to \infty} x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1,
\]

\[
\psi(x + 1) = \psi(x) + 1/x,
\]

\[
\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \cdots \quad (x \to \infty),
\]

\[
\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty e^{-xt} t^n \frac{dt}{1 - e^{-t}} = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x + k)^{n+1}},
\]

\[
\frac{1}{x} + \frac{1}{2x^2} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}.
\]

The formulas (2.1)–(2.7) can be found in [1], while (2.8) and corresponding rational bounds for \( \psi^{(n)} \) with \( n \geq 2 \) are given in [2, 10]. The following two lemmas present inequalities for the psi function.

**Lemma 2.2.** Let \( t \geq 3 \) be a real number and let \( a = 1 - 1/\Gamma(t) \). Then we have for all real numbers \( x \geq 1 \):

\[
0 < ax^{t-1} + (ax^t - 1)\psi(tx).
\]

**Proof.** We denote the expression on the right-hand side of (2.9) by \( f(x) \). Differentiation gives

\[
xf'(x) = g(x) + a(t - 1)x^{t-1} + atx^{t+1}\psi'(tx),
\]

where \( g(x) = axt^\psi(tx) - tx^\psi(tx) \). Since \( \psi \) is positive on \((1.461 \ldots, \infty)\), \( x \geq 1 \) and \( a \geq 1/2 \) imply \( g(x)/(tx) \geq \psi(tx)/2 - \psi'(tx) \). Since \( \psi \) and \(-\psi'\) are strictly increasing on \((0, \infty)\), we obtain

\[
\frac{g(x)}{tx} \geq \frac{1}{2} \psi(3) - \psi'(3) = 0.066 \ldots
\]
From (2.10) and (2.11) we conclude that \( f'(x) > 0 \) for \( x \geq 1 \). Hence we have
\[
 f(x) \geq f(1) = a + (a - 1)\psi(t) = \frac{h(t)}{\Gamma(t)}, \tag{2.12}
\]
where \( h(t) = \Gamma(t) - \psi(t) - 1. \) Differentiation yields \( h'(t) = \Gamma'(t) - \psi'(t) \) and \( h''(t) = \Gamma''(t) - \psi''(t). \) Since \( \Gamma'' \) and \(-\psi''\) are positive on \((0, \infty)\), we obtain for \( t \geq 3: h'(t) \geq h'(3) = 1.450 \ldots \) and \( h(t) \geq h(3) = 0.077 \ldots \), so that (2.12) implies that \( f \) is positive on \([1, \infty)\).

**Lemma 2.3.** Let \( n \geq 3 \) be an integer and let \( a = 1 - 1/\Gamma(n) \). Then we have for all real numbers \( x_i \geq 1 \) (\( i = 1, \ldots, n \)):
\[
 0 < \psi \left( \sum_{i=1}^{n} x_i \right) \left[ a \prod_{i=1}^{n} x_i - 1 \right] + a \left( \max_{1 \leq i < n} x_i \right)^{-1} \prod_{i=1}^{n} x_i.
\]
**Proof.** We may assume that \( x_1 \geq \cdots \geq x_n \geq 1 \). Let
\[
f(x_1, \ldots, x_n) = \psi \left( \sum_{i=1}^{n} x_i \right) \left[ a \prod_{i=1}^{n} x_i - 1 \right] + a \prod_{i=1}^{n} x_i
\]
and \( f_q(x) = f(x, x, x, \ldots, x, x_q, \ldots, x_n) \), where \( x > 0 \) and \( q \in \{1, \ldots, n-1\} \). We prove that \( f_q \) is increasing on \([x_{q+1}, \infty)\). Let \( x \geq x_{q+1} \) and \( y = qx + \sum_{i=q+1}^{n} x_i \geq n \). Differentiation gives
\[
\frac{1}{q} f_q'(x) = \psi(y) \left[ a x^q \prod_{i=q+1}^{n} x_i - 1 \right] + \psi(y) a x^{q-1} \prod_{i=q+1}^{n} x_i + a \left( 1 - 1/q \right) x^{q-2} \prod_{i=q+1}^{n} x_i.
\]
Since \( x^q \prod_{i=q+1}^{n} x_i \geq 1, a > 0, \psi(y) > 0 \), and \( \psi'(y) > 0 \), we obtain
\[
\frac{1}{q} f_q'(x) \geq (a - 1)\psi'(y) + a \psi(y) = g(y), \text{ say.}
\]
The functions \((a - 1)\psi'\) and \(a \psi\) are strictly increasing on \((0, \infty)\), so that we get
\[
\Gamma(n) g(y) \geq \Gamma(n) g(n) = \psi(n) \Gamma(n) - 1 - \psi'(n) = h(n), \text{ say.}
\]
Since \( h'(n) = \psi(n) \Gamma'(n) - 1 + \psi'(n) \Gamma'(n) - \psi''(n) > 0 \) for \( n \geq 3 \), we obtain
\[
 h(n) \geq h(3) = 0.527 \ldots .
\]
This implies that \( f_q'(x) > 0 \) for \( x \geq x_{q+1} \). Thus we get
\[
f(x_1, \ldots, x_n) = f_1(x_1) \geq f_2(x_2) = f_2(x_2) \geq f_2(x_3) \geq \cdots \geq f_{n-1}(x_n)
\]
\[
= a x_n^{n-1} + (ax_n^{n-1} - 1) \psi(nx_n).
\]
Applying Lemma 2.2 we conclude that \( f(x_1, \ldots, x_n) > 0 \).
Further, we need the following monotonicity theorem.

**Lemma 2.4.** Let \( a > 1 \) be a real number. The function

\[
\phi_a(x) = a(x - 1/2) \log x - (ax - 1/2) \log (ax) - a \log \Gamma(x) + \log \Gamma(ax)
\]

is strictly increasing on \((0, \infty)\) with \( \lim_{x \to \infty} \phi_a(x) = -\frac{1}{2}(a - 1) \log (2\pi) \).

**Proof.** Let \( x > 0 \). Differentiation gives

\[
x \phi_a'(x) = -\frac{a - 1}{2} - ax \log a - ax \psi(x) + ax \psi(ax) = p_a(x), \quad \text{say.} \tag{2.14}
\]

Further, we get

\[
\frac{1}{a} p_a'(x) = -\log a - \psi(x) + \psi(ax) - x \psi'(x) + ax \psi'(ax) \tag{2.15}
\]

and

\[
x \frac{1}{a} p_a''(x) = q(ax) - q(x), \tag{2.16}
\]

where \( q(x) = 2x \psi'(x) + x^2 \psi''(x) \). Next, we prove that \( q \) is strictly increasing on \((0, \infty)\). We obtain

\[
\frac{1}{x^2} q'(x) = \frac{2}{x} \psi'(x) + \frac{4}{x} \psi''(x) + \psi'''(x).
\]

Using the integral formulas (2.7) and

\[
\frac{1}{x^n} = \frac{1}{(n - 1)!} \int_0^\infty e^{-xt} t^{n-1} \, dt \quad (x > 0; n = 1, 2, \ldots),
\]

and the convolution theorem for Laplace transforms, we get

\[
\frac{1}{x^2} q'(x) = \int_0^\infty e^{-xt} \Lambda(t) \, dt, \tag{2.17}
\]

where

\[
\Lambda(t) = 2t \int_0^t \frac{s}{1 - e^{-s}} \, ds - 6 \int_0^t \frac{s^2}{1 - e^{-s}} \, ds + \frac{t^3}{1 - e^{-t}}.
\]

Let \( t > 0 \). Then we obtain

\[
\Lambda'(t) = 2 \int_0^t \frac{s}{1 - e^{-s}} \, ds - \left( \frac{t}{1 - e^{-t}} \right)^2 \left[ 1 + e^{-t}(t - 1) \right].
\]
and
\[
\frac{(1 - e^{-1})^3}{t^2} e^{2t} \Lambda''(t) = 2 + t + (t - 2)e^t = \sum_{k=3}^{\infty} \frac{k - 2}{k!} t^k > 0.
\]
Since \( \Lambda(0) = \Lambda'(0) = 0 \), we get \( \Lambda(t) > 0 \) for \( t > 0 \). From (2.17) we conclude that \( q \) is strictly increasing on \((0, \infty)\), so that (2.16) implies \( p_a''(x) > 0 \) for \( x > 0 \). Using the asymptotic expansion (2.6) and the limit relation \( \lim_{x \to \infty} x \psi'(x) = 1 \), we conclude from (2.14) and (2.15) that \( \lim_{x \to \infty} p_a(x) = \lim_{x \to \infty} p'_a(x) = 0 \). Thus \( p_a \) is positive on \((0, \infty)\). From (2.14) we obtain that \( p_a \) is strictly increasing on \((0, \infty)\).

The asymptotic formula (2.2) implies \( \lim_{x \to \infty} \phi'_a(x) = -\frac{1}{2}(a - 1) \log(2\pi) \).

### 3. Main results

We are now in a position to prove the inequalities for the beta function that we announced in Section 1. Our first theorem provides a generalisation of the double-inequality (1.2).

**Theorem 3.1.** Let \( n \geq 3 \) be an integer. Then we have for all real numbers \( x_i \geq 1 \) \((i = 1, \ldots, n)\):
\[
0 < \frac{1}{\prod_{i=1}^{n} x_i} - B(x_1, \ldots, x_n) \leq 1 - \frac{1}{(n - 1)!}. \tag{3.1}
\]
Both bounds are best possible.

**Proof.** The first inequality of (3.1) is equivalent to
\[
0 < \log \Gamma(x_1 + \cdots + x_n) - \sum_{i=1}^{n} \log \Gamma(x_i + 1). \tag{3.2}
\]
To prove (3.2) we may assume that \( x_1 \geq \cdots \geq x_n \geq 1 \). We denote the right-hand side of (3.2) by \( f(x_1, \ldots, x_n) \). Further, let \( q \in \{1, \ldots, n - 1\} \), \( x \geq x_{q+1} \), and
\[
f_q(x) = f(x, x, \ldots, x, x_{q+1}, \ldots, x_n) = \log \Gamma \left( qx + \sum_{i=q+1}^{n} x_i \right) - q \log \Gamma(x + 1) - \sum_{i=q+1}^{n} \log \Gamma(x_i + 1).
\]
Since \( \psi \) is strictly increasing on \((0, \infty)\), we get
\[
\frac{1}{q} f'_q(x) = \psi \left( qx + \sum_{i=q+1}^{n} x_i \right) - \psi(x + 1) > 0,
\]
so that \( f_q \) is strictly increasing on \([x_{q+1}, \infty)\). This implies

\[
f(x_1, \ldots, x_n) = f_1(x_1) \geq f_1(x_2) = f_2(x_2) \geq \cdots \geq f_{n-1}(x_{n-1}) = \log \Gamma(nx_n) - n \log \Gamma(x_n + 1).
\] (3.3)

Let \( g(x) = \log \Gamma(nx) - n \log \Gamma(x + 1) \). Then we get for \( x \geq 1 \):

\[
g'(x)/n = \psi(nx) - \psi(x + 1) > 0
\]

and

\[
g(x) \geq g(1) = \log \Gamma(n) \geq \log \Gamma(3) = \log 2.
\] (3.4)

From (3.3) and (3.4) we conclude that (3.2) is valid.

Using the asymptotic formula (2.1) we obtain

\[
\lim_{x \to \infty} \left( \frac{1}{x^n} - B(x, \ldots, x) \right) = \lim_{x \to \infty} \left( \frac{1}{x^n} - \left( \frac{\Gamma(x)}{\Gamma(nx)} \right)^n \right) = 0,
\]

which implies that in (3.1) the lower bound 0 cannot be replaced by a larger constant.

Let \( a = 1 - 1/(n - 1)! \). To prove the right-hand side of (3.1) we have to show that

\[
0 \leq \frac{\Gamma}{n} \left( \sum_{i=1}^{n} x_i \right) \left[ a \prod_{i=1}^{n} x_i - 1 \right] + \frac{\Gamma}{n} \prod_{i=1}^{n} \Gamma(x_i + 1) = u(x_1, \ldots, x_n), \quad \text{say}.
\]

Let \( q \in \{1, \ldots, n-1\}, x_1 \geq \cdots \geq x_n \geq 1, \) and

\[
u_q(x) = u(x, \ldots, x, x_{q+1}, \ldots, x_n)
\]

\[
= \Gamma \left( qx + \sum_{i=q+1}^{n} x_i \right) \left[ ax^q \prod_{i=q+1}^{n} x_i - 1 \right] + \left( \Gamma(x + 1) \right)^q \prod_{i=q+1}^{n} \Gamma(x_i + 1).
\]

We set \( y = qx + \sum_{i=q+1}^{n} x_i \) and apply Lemma 2.3. Then we get for \( x \geq x_{q+1} \):

\[
(q \Gamma(y))^{-1} \nu_q(x) = \psi(y) \left[ ax^q \prod_{i=q+1}^{n} x_i - 1 \right] + ax^{q-1} \prod_{i=q+1}^{n} x_i
\]

\[
+ \left( \Gamma(x + 1) \right)^q \psi(x + 1) (\Gamma(y))^{-1} \prod_{i=q+1}^{n} \Gamma(x_i + 1) > 0.
\]

Hence \( u_q \) is strictly increasing on \([x_{q+1}, \infty)\). This implies

\[
u(x_1, \ldots, x_n) = u_1(x_1) \geq u_1(x_2) = u_2(x_2) \geq \cdots \geq u_{n-1}(x_{n-1})
\]

\[
= (ax_n^n - 1) \Gamma(nx_n) + (\Gamma(x_n + 1))^n.
\] (3.5)
Let \( v(x) = (ax^n - 1)\Gamma(nx) + \left( \Gamma(x + 1) \right)^n \). Then we have
\[
\frac{v'(x)}{n\Gamma(nx)} = ax^{n-1} + (ax^n - 1)\psi(nx) + \frac{\left( \Gamma(x + 1) \right)^n \psi(x + 1)}{\Gamma(nx)}.
\]
From Lemma 2.2 we conclude that \( v \) is strictly increasing on \([1, \infty)\). Thus
\[
v(x) \geq v(1) = 0 \quad \text{for} \quad x \geq 1,
\]
so that (3.5) yields \( u(x_1, \ldots, x_n) \geq 0 \).

If \( x_1 = \cdots = x_n = 1 \), then the second inequality of (3.1) holds with equality. This implies that the upper bound \( 1 - 1/(n - 1)! \) is sharp.

**Remark.** The inequalities (3.1) are not valid for all positive real numbers \( x_i \) \((i = 1, \ldots, n)\). More precisely: there do not exist constants \( c_1(n) \) and \( c_2(n) \) such that
\[
c_1(n) \leq \frac{1}{\prod_{i=1}^n x_i} - B(x_1, \ldots, x_n) \leq c_2(n)
\]
holds for all \( x_i > 0 \) \((i = 1, \ldots, n; n \geq 2)\). Indeed, if we set \( x_1 = \cdots = x_{n-1} = x > 0 \) and \( x_n = y > 1 \), then the left-hand side of (3.6) yields
\[
x^{n-1} y c_1(n) \leq 1 - \frac{\left( \Gamma(x + 1) \right)^n \Gamma(y + 1)}{\Gamma(n - 1)x + y}.
\]
We let \( x \) tend to 0 and obtain the incorrect inequality \( 0 \leq 1 - \Gamma(y + 1)/\Gamma(y) = 1 - y \).

And, if we set \( x_1 = \cdots = x_n = x > 0 \), then the right-hand side of (3.6) gives
\[
\frac{1}{x^n} - \frac{\left( \Gamma(x) \right)^n}{\Gamma(nx)} = \frac{\Gamma(nx + 1) - n x \left( \Gamma(x + 1) \right)^n}{x^n \Gamma(nx + 1)} \leq c_2(n).
\]
This is false, since the term on the left-hand side tends to \( \infty \), if we let \( x \) tend to 0.

The next theorem provides sharp upper and lower bounds for \( B(x_1, \ldots, x_n) \), which are valid in \([c, \infty)^n\), where \( c > 0 \) is a fixed real number.

**Theorem 3.2.** Let \( c > 0 \) be a real number and let \( n \geq 2 \) be an integer. Then we have for all real numbers \( x_i \geq c \) \((i = 1, \ldots, n)\):
\[
\alpha_n(c) \prod_{i=1}^n x_i^{-1/2 + x_i} \left( \sum_{i=1}^n x_i ight)^{-1/2 + \sum_{i=1}^n x_i} < B(x_1, \ldots, x_n) \leq \beta_n(c) \prod_{i=1}^n x_i^{-1/2 + x_i} \left( \sum_{i=1}^n x_i ight)^{-1/2 + \sum_{i=1}^n x_i},
\]
with the best possible constants
\[
\alpha_n(c) = (2\pi)^{(n-1)/2} \quad \text{and} \quad \beta_n(c) = n^{nc-1/2} c^{(n-1)/2} \frac{\Gamma(c)^n}{\Gamma(nc)}.
\]
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**Proof.** Let \( x > 0 \) and \( x_i > 0 \) \( (i = 1, \ldots, n) \) be real numbers and let \( q \in \{1, \ldots, n - 1\} \). We define

\[
 f(x_1, \ldots, x_n) = \sum_{i=1}^{n} (x_i - 1/2) \log x_i - \left( \sum_{i=1}^{n} x_i - 1/2 \right) \log \left( \sum_{i=1}^{n} x_i \right)
 - \sum_{i=1}^{n} \log \Gamma(x_i) + \log \Gamma \left( \sum_{i=1}^{n} x_i \right)
\]

and

\[
 f_q(x) = f(x, \ldots, x, x_{q+1}, \ldots, x_n)
 = q \left( x - \frac{1}{2} \right) \log x + \sum_{i=q+1}^{n} (x_i - 1/2) \log x_i
 - \left( qx + \sum_{i=q+1}^{n} x_i - 1/2 \right) \log \left( qx + \sum_{i=q+1}^{n} x_i \right) - q \log \Gamma(x)
 - \sum_{i=q+1}^{n} \log \Gamma(x_i) + \log \Gamma \left( qx + \sum_{i=q+1}^{n} x_i \right).\]

Then we get \( f_q'(x)/q = g(x) - g(y) \), where \( g(z) = \log z - 1/(2z) - \psi(z) \) and \( y = qx + \sum_{i=q+1}^{n} x_i \). The left-hand side of \( (2.8) \) implies \( g'(z) = 1/z + 1/(2z^2) - \psi'(z) < 0 \) for \( z > 0 \). Hence we conclude from \( y > x \) that \( g(y) < g(x) \). This implies that \( f_q \) is strictly increasing on \((0, \infty)\).

To prove the right-hand inequality of \( (3.7) \) with \( \beta_n(c) \) as defined in \( (3.8) \), we assume that \( x_1 \geq \cdots \geq x_n \geq c \). Then we obtain

\[
 f(x_1, \ldots, x_n) = f_1(x_1) \geq f_2(x_2) \geq f_3(x_3) \geq \cdots \geq f_{n-1}(x_n) = \phi_n(x_n), \tag{3.9}
\]

where \( \phi_n \) is defined in \( (2.13) \). From Lemma 2.4 we get

\[
 \phi_n(x_n) \geq \phi_n(c) = - \log \beta_n(c), \tag{3.10}
\]

so that \( (3.9) \) and \( (3.10) \) lead to

\[
 f(x_1, \ldots, x_n) \geq - \log \beta_n(c), \tag{3.11}
\]

which is equivalent to the second inequality of \( (3.7) \). Moreover, since \( f_q \) and \( \phi_n \) are strictly monotonic, we conclude that the sign of equality holds in \( (3.11) \) if and only if \( x_1 = \cdots = x_n = c \).

To prove the left-hand side of \( (3.7) \) with \( \alpha_n(c) = (2\pi)^{(n-1)/2} \) we suppose that \( c \leq x_1 \leq \cdots \leq x_n \). The monotonicity of \( f_q \) and Lemma 2.4 lead to

\[
 f(x_1, \ldots, x_n) = f_1(x_1) \leq f_2(x_2) \leq f_3(x_3) \leq \cdots \leq f_{n-1}(x_n)
\]
which leads to the first inequality of (3.7) with \( \alpha_x(c) = (2\pi)^{(n-1)/2} \).

Conversely, we assume that the left-hand inequality of (3.7) is valid for all \( x_i \geq c \) \((i = 1, \ldots, n)\). Then we set \( x_1 = \cdots = x_n = x > 0 \) and obtain \( \alpha_x(c) < e^{-\phi(x)} \).

Applying Lemma 2.4 we get \( \alpha_x(c) \leq \lim_{c \to \infty} e^{-\phi(x)} = (2\pi)^{(n-1)/2} \). Thus in (3.7) the factor \( \alpha_x(c) = (2\pi)^{(n-1)/2} \) cannot be replaced by a larger constant.

If a function \( f \) satisfies the inequality \( f(\delta x_1, \ldots, \delta x_n) \leq \delta f(x_1, \ldots, x_n) \) for all \( x_i > 0 \) \((i = 1, \ldots, n)\) and \( \delta \in (0, 1) \), then \( f \) is said to be starshaped on \( \mathbb{R}_+^n \). Interesting properties of these functions can be found in [6, 7]. As an immediate consequence of the following theorem we obtain that the beta function is not starshaped on \( \mathbb{R}_+^n \).

**Theorem 3.3.** Let \( \mu \) and \( \nu \) be real numbers with \( \mu > \nu > 0 \) and let \( n \geq 2 \) be an integer. Then we have for all real numbers \( x_i > 0 \) \((i = 1, \ldots, n)\):

\[
0 < \frac{B(x_1, x_2, \ldots, x_n)}{B(x_1, x_2, \ldots, x_n)} < \left( \frac{\nu}{\mu} \right)^{n-1}.
\]

(3.12)

Both bounds are best possible.

**Proof.** To establish the second inequality of (3.12) it suffices to show that the function \( f(t) = t^{n-1} B(t x_1, \ldots, t x_n) \) is strictly decreasing on \((0, \infty)\). Let \( t > 0 \). Differentiation yields

\[
\frac{t}{f(t)} f'(t) = n - 1 + \sum_{i=1}^n t x_i \psi(t x_i) - \psi \left( \sum_{i=1}^n t x_i \right) \sum_{i=1}^n t x_i.
\]

(3.13)

We set \( y_i = t x_i > 0 \) \((i = 1, \ldots, n)\) and define

\[
g(y_1, \ldots, y_n) = \psi \left( \sum_{i=1}^n y_i \right) \sum_{i=1}^n y_i - \sum_{i=1}^n y_i \psi(y_i).
\]

In order to prove

\[
g(y_1, \ldots, y_n) > n - 1
\]

we assume that \( y_1 \geq \cdots \geq y_n > 0 \). Let \( q \in [1, \ldots, n-1], y > 0, \) and

\[
g_q(y) = g(y, \ldots, y, y_{q+1}, \ldots, y_n)
\]

\[
= \psi \left( q y + \sum_{i=q+1}^n y_i \right) \left( q y + \sum_{i=q+1}^n y_i \right) - q y \psi(y) - \sum_{i=q+1}^n y_i \psi(y_i).
\]
Then we get
\[ g_q'(y)/q = h(z) - h(y), \]  
(3.15)
where
\[ h(x) = \psi(x) + x\psi'(x) \quad \text{and} \quad z = qy + \sum_{i=q+1}^{n} y_i. \]  
(3.16)

Using the series representation (2.7) we obtain
\[ h'(x) = 2\psi'(x) + x\psi''(x) = 2 \sum_{k=1}^{\infty} \frac{k}{(x+k)^2} > 0. \]

Since \( z > y \), we get \( h(z) > h(y) \), so that (3.15) implies that \( g_q \) is strictly increasing on \((0, \infty)\). Hence we have
\[ g(y_1, \ldots, y_n) = g_1(y_1) \geq g_2(y_2) \geq \cdots \geq g_{n-1}(y_n) \]
\[ = ny_n[\psi(ny_n) - \psi(y_n)]. \]  
(3.17)

Let
\[ \omega(y) = ny[\psi(ny) - \psi(y)]. \]  
(3.18)

Then \( \omega'(y)/n = h(ny) - h(y) \), where \( h \) is defined in (3.16). Since \( h \) is strictly increasing on \((0, \infty)\), we obtain \( \omega'(y) > 0 \) and
\[ \omega(y) > \lim_{t \to 0} \omega(t) \quad (y > 0). \]  
(3.19)

The recurrence formula (2.5) implies
\[ \lim_{t \to 0} \omega(t) = n - 1, \]  
(3.20)
so that (3.17)–(3.20) lead to (3.14). From (3.13) and (3.14) we conclude that \( f \) is strictly decreasing on \((0, \infty)\).

If we set \( x_1 = \cdots = x_n = x > 0 \), then we have
\[ \frac{B(\mu x_1, \ldots, \mu x_n)}{B(v x_1, \ldots, v x_n)} = \left( \frac{\Gamma(\mu x + 1)}{\Gamma(\mu x + 1)} \right)^n \left( \frac{\Gamma(nv x + 1)}{\Gamma(nv x + 1)} \right)^{n-1} \left( \frac{\nu}{\mu} \right). \]

This implies
\[ \lim_{x \to 0} \frac{B(\mu x_1, \ldots, \mu x_n)}{B(v x_1, \ldots, v x_n)} = \left( \frac{\nu}{\mu} \right)^{n-1}. \]  
(3.21)
And, if we put $x_1 = x > 0$, $x_2 = \cdots = x_n = 1$, then we get

$$
\frac{B(\mu x_1, \ldots, \mu x_n)}{B(\nu x_1, \ldots, \nu x_n)} = \frac{\Gamma(\mu x)}{\Gamma(\mu x + (n-1)\nu)} \frac{\Gamma(\nu x + (n-1)\mu)}{\Gamma(\nu x)} \frac{x^{(\nu-1)(\mu-\nu)}}{\mu^{\nu-1}}.
$$

(3.22)

From (2.4) and (3.22) we obtain

$$
\lim_{x \to \infty} \frac{B(\mu x_1, \ldots, \mu x_n)}{B(\nu x_1, \ldots, \nu x_n)} = 0.
$$

(3.23)

The limit relations (3.21) and (3.23) imply that the bounds given in (3.12) are best possible.

A function $f : \mathbb{R}_+^n \to \mathbb{R}$ is called midconvex (or Jensen-convex) if we have for all $x_i, y_i > 0$ ($i = 1, \ldots, n$):

$$
f\left((x_1 + y_1)/2, \ldots, (x_n + y_n)/2\right) \leq \frac{1}{2}(f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n)).
$$

(3.24)

It is known that a continuous midconvex function is also convex; see [17]. We now prove that $f(x_1, \ldots, x_n) = \log B(x_1, \ldots, x_n)$ satisfies (3.24), which implies that the beta function is log-convex on $\mathbb{R}_+^n$. This extends a result given in [9], where a proof for the log-convexity of $B(x, y)$ is given.

**Theorem 3.4.** Let $n \geq 2$ be an integer. Then we have for all real numbers with $x_i > 0$ and $y_i > 0$ ($i = 1, \ldots, n$):

$$
0 < \frac{B((x_1 + y_1)/2, \ldots, (x_n + y_n)/2)}{\sqrt{B(x_1, \ldots, x_n)B(y_1, \ldots, y_n)}} \leq 1.
$$

(3.25)

Both bounds are best possible.

**Proof.** The Cauchy-Schwarz inequality for integrals yields

$$
(B(x_1 + y_1, x_2 + y_2))^2 = \left(\int_0^1 t^{x_1-1/2}(1-t)^{y_1-1/2} t^{x_2-1/2}(1-t)^{y_2-1/2} dt\right)^2 \\
\leq \int_0^1 t^{x_1-1}(1-t)^{x_2-1} dt \int_0^1 t^{y_1-1}(1-t)^{y_2-1} dt \\
= B(2x_1, 2x_2)B(2y_1, 2y_2).
$$

(3.26)

Using the representation

$$
B(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} B\left(\sum_{j=i+1}^{n} x_j, x_{i+1}\right)
$$
and (3.26) we obtain
\[
(B(x_1 + y_1, \ldots, x_n + y_n))^2 = \prod_{i=1}^{n-1} \left[ B \left( \sum_{j=1}^{i} x_j + \sum_{j=1}^{i} y_j, x_{i+1} + y_{i+1} \right) \right]^2 \\
\leq \prod_{i=1}^{n-1} \left[ B \left( 2 \sum_{j=1}^{i} x_j, 2x_{i+1} \right) \left( 2 \sum_{j=1}^{i} y_j, 2y_{i+1} \right) \right] \\
= B(2x_1, \ldots, 2x_n)B(2y_1, \ldots, 2y_n).
\]

This proves the right-hand side of (3.25). If we set \(x_i = y_i = z > 0\) \((i = 1, \ldots, n)\), then equality holds in the second inequality of (3.25). Further, we have
\[
\lim_{x_i \to 0} \frac{(B((x_1 + y_1)/2, \ldots, (x_n + y_n)/2))^2}{B(x_1, \ldots, x_n)B(y_1, \ldots, y_n)} = 0,
\]
so that in (3.25) the lower bound 0 cannot be improved.

A function \(f : \mathbb{R}_+^n \to \mathbb{R}\) is said to be subadditive if the inequality
\[
f(x_1 + y_1, \ldots, x_n + y_n) \leq f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n) \quad (3.27)
\]
holds for all \(x_i, y_i > 0\) \((i = 1, \ldots, n)\). Subadditive functions play a role in the theory of differential equations, in the theory of convex bodies, and also in the theory of semi-groups; see [18]. From the following theorem we conclude that for all real numbers \(c > 0\) the function \((x_1, \ldots, x_n) \mapsto (B(x_1, \ldots, x_n))^c\) is subadditive on \(\mathbb{R}_+^n\).

**Theorem 3.5.** Let \(c > 0\) be a real number and let \(n \geq 2\) be an integer. Then we have for all real numbers \(x_i > 0\) and \(y_i > 0\) \((i = 1, \ldots, n)\):
\[
0 < \frac{(B(x_1 + y_1, \ldots, x_n + y_n))^c}{(B(x_1, \ldots, x_n))^c + (B(y_1, \ldots, y_n))^c} < 2^{-c(n-1)-1}. \quad (3.28)
\]
Both bounds are best possible.

**Proof.** To prove the second inequality of (3.28) we apply Theorem 3.4, the arithmetic mean-geometric mean inequality, and Theorem 3.3 (with \(\mu = 2, v = 1\)). Then we get
\[
(B(x_1 + y_1, \ldots, x_n + y_n))^c \leq [B(2x_1, \ldots, 2x_n)B(2y_1, \ldots, 2y_n)]^{c/2} \\
\leq \frac{1}{2} \left[ (B(2x_1, \ldots, 2x_n))^c + (B(2y_1, \ldots, 2y_n))^c \right] \\
< 2^{-c(n-1)-1} \left[ (B(x_1, \ldots, x_n))^c + (B(y_1, \ldots, y_n))^c \right].
\]
It remains to show that the bounds given in (3.28) are sharp. First, we set $x_i = y_i = z > 0$ ($i = 1, \ldots, n$). The duplication formula (2.3) leads to, say,

$$
\frac{(B(x_1, y_1, \ldots, x_n, y_n))^c}{(B(x_1, \ldots, x_n))^c + (B(y_1, \ldots, y_n))^c} = \frac{1}{2} \left( \frac{1}{2\sqrt{\pi}} \right)^c (\frac{(\Gamma(z+1/2))^n}{\Gamma(nz+1/2)})^c = f(z).
$$

Since $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$
\lim_{z \to 0} f(z) = 2^{-c(n-1)-1}.
$$

(3.29)

And using (2.1) we get

$$
\lim_{z \to \infty} f(z) = 0.
$$

(3.30)

From (3.29) and (3.30) we conclude that both bounds in (3.28) are best possible.

**Remark.** A multiplicative analogue of the definition (3.27) is given by

$$
f(x_1, y_1, \ldots, x_n, y_n) \leq f(x_1, \ldots, x_n) f(y_1, \ldots, y_n).
$$

(3.31)

If $f$ satisfies (3.31) for all $x_i, y_i > 0$ ($i = 1, \ldots, n$), then $f$ is said to be submultiplicative on $\mathbb{R}^n_+$. These functions have applications in functional analysis and group theory; see [11, 14]. If (3.31) holds with “≥” instead of “≤”, then $f$ is called supermultiplicative. Let $n \geq 2$. We set $x_i = 1$ (2 ≤ $i$ ≤ $n$) and $y_i = 1$ (1 ≤ $i$ ≤ $n$; $i \neq 2$). Then we obtain, say,

$$
\frac{B(x_1, y_1, \ldots, x_n, y_n)}{B(x_1, \ldots, x_n) B(y_1, \ldots, y_n)} = \frac{\Gamma(x_1 + n - 1)}{\Gamma(x_1 + y_2 + n - 2)} x_1^{y_2-1} \Gamma(y_2 + n - 1)x_1^{1-y_2}
$$

$$
= \sigma(x_1).
$$

Applying (2.4) we get: if $y_2 > 1$, then $\lim_{x_1 \to \infty} \sigma(x_1) = 0$; and, if $y_2 \in (0, 1)$, then we have $\lim_{x_1 \to \infty} \sigma(x_1) = \infty$. This implies that $(x_1, \ldots, x_n) \mapsto B(x_1, \ldots, x_n)$ is neither submultiplicative nor supermultiplicative on $\mathbb{R}^n_+$.

**References**


Inequalities for the beta function of $n$ variables