The essentials of, and for, mathematics

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The first qualification for judging a piece of workmanship from a corkscrew to a cathedral is to know what *it is — what it was intended to do and how it is meant to be used.* C.S. Lewis [6].

How do we approach the teaching of mathematics?[1] How do we approach reviewing the present curriculum? In this paper we attempt to give a framework for answering these questions and although these questions are addressed in the context of high schools, the ideas in this article are equally applicable to the university context. We describe attitudes rather than content. The author believes that being constantly aware of the context in which students will operate will assist in revising the design and lead to a syllabus and recommendations that are not only appropriate to our times, but also can be readily and effectively implemented.

1 Introduction

Sixty years ago, Courant and Robbins wrote *What is mathematics? An elementary approach to ideas and methods* [1]. From today's viewpoint this seems an incredibly narrow view of mathematics, because mathematics is now employed, and indeed effectively employed, in so many more fields than they ever envisaged.

Why the change? Let me present some slogans (of my own)

Mathematics has no end, and many ends.

Many, perhaps most, mathematicians will feel uncomfortable with such a paradox. Shortly we will turn to the paradox but remember that major advances in mathematics have come from paradoxes. Obvious examples are the discovery of irrational numbers by Hippasus (see, for example my [2]), the problems of infinitesimals in the nineteenth century and Cantor's paradoxes in set theory. From these three alone came, eventually, Cartesian geometry, the modern foundations of analysis and the continuing vitality of set theory and mathematical logic. Although, one must admit, the Cantor's paradoxes are still not satisfactorily settled for everyone.

Let us return to our paradox: *Mathematics has no end.*

This is obviously true to mathematicians but, to the surprise of many mathematicians, almost incomprehensible to the general public. Any mathematician knows there are always unsolved problems, and new areas of mathematics itself to be developed. The general public thinks, far too often, that mathematics is a finished collection of rules and calculations.

More than that, the notions (the plural is intentional) of infinity permeate large amounts of mathematics. In particular students are introduced to non-terminating decimals early on.

Now there are at least two, very different, aspects of infinity to consider here.

- The possibility of proceeding indefinitely – as in counting $1, 2, 3, \ldots$. and
- Infinite repetition of the *same* calculation produces the same result each time.

[1] This paper was originally written in response to an invitation from the Victorian Curriculum and Assessment Authority with respect to its work on developing a *Framework of Essential Learning.* The views expressed in the paper are those of the author and do not necessarily represent the views of the Authority. I am particularly grateful to Barry Jay, Tim Brook and Eve Wirth for valuable suggestions and to a number of other people who have read this article.

The former opens up possibilities, the latter gives people confidence in mathematics.

2 Mathematics as a human activity

Let us now turn to the ends of mathematics; that is to say, 'ends' in the sense of 'purposes'. Apart from pure mathematicians, the world in general believes that mathematics is indeed the servant of the sciences. It produces tangible and verifiable results – in many different fields. So the usefulness of mathematics is a major consideration for most people who employ mathematics. And this is the principal motivation for the majority of those people who want to study mathematics, apart (again) perhaps from pure mathematicians. But let us not forget that

Mathematics is a human activity.

This goes without question. What does not go without question is the *content* of mathematics. I would argue that we cannot separate the content from the process.

If we go back to Aristotle we find him looking at the organization of knowledge. I see Aristotle as one who uses logic as a means of rearranging facts or information. Thereby we can find the information we really want to have. Mathematics does this rearranging *par excellence.*

Over the centuries mathematics has been organized and formalized so that people now look at mathematics as *the* way to find the facts, though, one must admit, these tend to be numerical facts.

Mathematics is certainly the key to science and to economics. It is almost unreasonably effective. As Wigner says in his much quoted article [9]: "The first point is that mathematical concepts turn up in entirely unexpected connections." Certainly mathematics is unexpectedly effective to many people.

Mathematics, despite all the hype about mathematics being timeless and even existing without us, is a human activity. The way we look at the world, the way we organize our world in order to comprehend it, the way we look at its structures – all of these stem from our human condition. This has only recently begun to be accepted (see my [2] and Lakoff and Nuñez's [5]) but note Keith Devlin's comment (in email, the original article is [3]):

I think . . . the attitude toward proof today is more realistic than it was when you and I were students [in the middle of the twentieth century], when diagrams were verboten, and where proofs were meant to be logically sound but unintelligible except after great effort.

Given that mathematics is a human activity it should be unsurprising that it changes over time – and so does its subject matter. (For an exquisite account of how the subject matter changes – and becomes more precise – see Lakatos [4].)

So, despite the conviction of some, I cannot believe that mathematics is absolute. There is no tangible evidence for a Platonic heaven, but the world in which we live continually throws up interesting challenges to which mathematics responds, and thereby is developed.

In teaching mathematics, therefore, I believe we are trying to open minds, not to close them. How do we do this?

3 Our context

First of all we have to consider the context. In teaching mathematics the subject matter makes sense to the student only if it relates to his or her previous experience. In ancient China mathematics was used primarily for astronomy and for taxes (see [7]). Therefore mathematics developed to support these interests: it was important to know the phases of the moon and the locations of the (known) planets, so mathematics developed interpolation techniques. (Interestingly this did not lead to the calculus, which arose first in the West and was only later taken into China.)

For taxes, other, different, kinds of mathematics were developed, leading to, amongst other things, the study of quadratic and higher degree polynomial equations.

In the West, as in the East, concerns of war also accelerated developments in science in general and mathematics in particular. In these developments mathematicians had to rely on their (human) experience.

One of the most spectacular developments in the West, although Euclid had tangled with it (see my [2], chapter 6), was Descartes's correlation of numbers with the (geometric) line, which eventually led to the development of the calculus. I could have used the word "identification" instead of "correlation" but I prefer not to, because a geometric line is *not* a set of numbers nor any sort of construct from numbers. Nevertheless, in order to work out what happens with lines it usually was, and is, much easier to deal with numbers than to think in terms of lines.

Descartes's view provides a metaphor.[2] That is to say, that the number line and the geometric line are not the same object, but that our view of the number line allows us to bring geometric lines into a context where we feel comfortable, and therefore find ourselves able to reason about such lines. The number line, beloved of many, is a metaphor, not a reality. But it works wonderfully well. Using the metaphor, Descartes succeeded in changing paradigms. He used a structure we know — that of the real numbers — to unlock the unknown, or at least largely unknown, world of parabolas and other conic sections and to go far beyond them.

In the same vein, our use of modelling, in particular mathematical modelling, has been immensely valuable in solving problems of physics, and the natural sciences, and even in meteorology. In recent centuries such modelling has expanded from looking at continuous structures: in fluid dynamics, for example, to discrete ones: graphs and networks in particular, and then even further to more complicated discrete structures such as cyphers and computer programs. These latter have meant that mathematicians have had to develop new mathematics. Such includes new parts of combinatorics and graph theory as well as of logic and set theory.

So theories of languages and strings have been developed. Such have not featured so much in school syllabuses but are omnipresent in daily life, for example, in newsagents where one finds unending successions of so-called "logic puzzle" magazines and now sudoku.[3] These involve logic and combinatorics at least. Last century such theories extended into the development of algorithms.

4 Computation and proof

Algorithms are now today's lifeblood, as functions were in the nineteenth and early twentieth centuries. Algorithms make us think of computations but, as we know to our cost as computer users, algorithms, such as the ones in the software that we use everyday, do not always produce the answer or behaviour that they should.

Along with means of computation one also needs means of proof. Together with each algorithm one should have a proof that the algorithm does indeed do what it is supposed to do. This is because there are two kinds of problems here. The first is the obvious one of human error. The second is that the specification for the computation, while being correct as a specification, may not be a specification for what was really desired to be computed.

In elementary algebra it is sometimes hard to see that the algorithm does indeed

[2] In this context see [5] for an extended discussion of how mathematics depends on metaphor.

[3] I am not suggesting that the theories of such objects should be in the high school syllabus but I am saying that students deal with the objects of these theories. Moreover, they are comfortable with these objects as the popularity of puzzles associated with them demonstrates.

give the correct result. E.g., if one takes the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for solving a quadratic equation of the form $ax^2 + bx + c = 0$, then it takes some work to show that the right hand side does indeed satisfy the equation.

In elementary geometry, using ruler and compasses, it is perhaps hard to distinguish computation, or equivalently, construction, from proof. Nevertheless both are necessary. When we look at the calculus, then finding the area under a (nice, simple, continuous) curve by a simple formula-driven computation really does need an explicit proof that it gives the right answer.

Thus we know that $\int x^2 dx = \frac{1}{3}x^3 + c$ by the algorithm that $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ provided $n \neq -1$. Nevertheless, proving that this step towards the result is correct, or to put it another way, that it actually leads us to the area, is non-trivial.

So computation and proof should be developed together. Otherwise, how does anyone see the need for a proof that the computation actually works?

There is also the question of why proofs are needed in mathematics in general. This is sometimes harder to see. Frank Harary gave a very nice exposition on this topic many years ago in Malaysia in 1974. I heard it, and I still remember and heed his instructions. The essence was as follows. Suppose you want to present to an audience a theorem that A happens under the hypothesis H.

- Give examples where A occurs.
- Give counterexamples where it does not occur.
- Prove the theorem under the hypothesis H.
- Give examples where the hypothesis H does not hold, and A does not occur.

Such a procedure also leads to better understanding, and surely understanding should accompany doing (in particular, calculating or computing). Proofs give understanding, or at least they should; computations give results.

5 The rôle of technology

At this point in our history we can use mechanical, or more frequently electronic, devices to perform repetitious tasks or to shortcut them. This has been the case since slide rules were introduced – some might even say, since the abacus was introduced. What is different now is that a multitude of tasks, identical or different, can be performed at, so to say, the touch of a button and very quickly. This change in quantity brings with it a change in quality.[4]

It is easy to deal well with single computations such as finding an n-th root. This is because we can give a clear and explicit process for finding such a root *and* a proof that it is correct. On the other hand we do not seem to be able to deal as well, or as adequately, with long sequences of computations. The very complexity of some computations is too much for us to grasp. We do not seem to be able to handle such sequences in a way that is satisfactory enough from a formal point of view (*cf.* Devlin's article [3]).

In order to equip the next generation adequately we need to give them sufficient understanding of the individual processes, and also sufficient understanding of sequences of processes, so that they can make informed judgments about the reliability of the software and hardware that they will employ. This cannot start too early.

As well as examples of their usefulness, examples can easily be given where there are problems: try taking the square root of a number over and over again on (different) hand-held calculators: usually twenty or so times will suffice to illustrate the problem.

Let us also be thoughtful about when it is appropriate to use computers or calculators. We use them when we do repetitious

[4]Compare the music of Philip Glass where repetition gives a different quality to his music.

work. When we need to do a calculation a thousand times, or even weekly or daily, it is foolish to use pen and paper: a computer or calculator is more reliable and less stressful. However, for a one-off, intricate calculation it will usually be easier to just do it, rather than to write, test and run a computer program for it.

This is one issue involving scale, here meaning the number of times we repeat a calculation.

6 The question of scale

One of the great virtues of mathematics is that, since mathematics gives us the logic of the world, we can use it for any scale of activity. Here "scale" does not only refer to physical size. It also includes complexity. Thus in training people to use and understand mathematics we should show them how we can shift from macrocosm to microcosm, or anywhere in between, and still be able to use our mathematical techniques. Likewise we can look at processes, and at pieces of software, in the same mathematical way.

We can consider processes and programs as structures. Within themselves they will have other, smaller structures. To describe these we need the *language* of mathematical structures. (See also below §7, p. 183.) I am not convinced that there is a great need for very formal proofs in the lower years of high school: in many cases a convincing argument will suffice. Nevertheless some basic facts about graphs, such as the size of a binary tree, could easily be introduced and established.

Understanding structures is something that can only be done by looking at them in different ways. One of the most productive is to look at them as quite coarse-grained. For example, in the context of logical arguments which, following Aristotle (see above §2, p. 178), are just as good objects to study as maps (in either obvious sense) and buildings: the structure of an argument only becomes clear when one removes all the superfluous material. For example, political arguments have all sorts of emotional (and, of course, political) overtones and embellishments.

In my local State (Victoria, Australia) in the *Curriculum and Standards Framework II* for Mathematics [8] (http://csf.vcaa.vic.edu.au/ma/lima----.htm), Section 5.2 MARSR502 in *Outcomes*) we read:

Make judgments about the quality of the reasoning in a mathematical argument expressed verbally or in symbolic form.

This is evident when the student is able to:

- *follow a short chain of reasoning from a general rule to a particular instance* (*e.g. the substitution of a number into the rule for a function and the subsequent arithmetic*)
- *follow a short chain of reasoning or from ideas accepted as being true to logical conclusion*(*s*) (*e.g. that the factors of rectangular and square numbers can be used to demonstrate the equivalence of simple algebraic expressions*).

One has to look at both logical arguments and other (mathematical) structures in the appropriate scale and neglect fine detail in structures in order to be able to handle them with just a few steps.[5] But what are the structures at which we should look?

7 What is it essential to study?

Any answers to this question from virtually any collection of at least two mathematicians will yield divergent answers. So let me first ask:

Why should we study what we do?

The items that we need to consider for our daily lives include taxes, gambling, building and constructing, reading

[5] Technically one could look at *homomorphisms* but without giving a formal definition. It is not clear that such a concept should be formally defined at this stage of schooling.

the newspaper, watching television, using computers and generally understanding our world. For these, many techniques spring to mind. Now consider the usefulness of such techniques. We need to know how to use calculators to a greater or lesser extent. We need to *understand* how they work to a greater, or more probably lesser, extent. (See above §4, p. 179.) So this should affect how we look at the *process* of calculation and not just its execution.

What kinds of things should we study?

In general in life we are confronted with problems that require solutions. For these solutions we need techniques, we need abstraction and, to reduce our efforts, we need reusability.

Not everyone can master the required abilities. So when we look at educating the whole population we need to give everyone general equipment, but we may choose to give a smaller fraction of the population the opportunity to study more advanced topics. Interestingly (as Peter Fensham revealed quite a time ago) although there is a smaller proportion of the (Australian) population undertaking specialist mathematics, nevertheless the *general* level of mathematical education has increased.

It is therefore important to balance the needs and abilities of the general population against the desires of professional mathematicians.

Getting the balance right

What then should we teach the general population — and what the potential professional mathematicians? It is important to provide for the latter since they will be the ones to lead and teach the next generation. For the general population, the relevance and accessibility of the mathematics taught will be paramount.

This creates a conflict for professional mathematicians that has been very noticeable in recent years. Some believe there is a canon of mathematics that everyone should learn. Nevertheless most professional mathematicians appear not to be dismayed by (or not aware of) the fact, noted above, that virtually any two mathematicians will disagree on the contents of the canon.

Students, on the other hand, are at the mercy of many other interests competing with mathematics. Moreover, there are far more such interests than there were when the teachers were trained. It will therefore be necessary to prioritize. This is something that was less urgent in the last century. It is of prime importance now.

So my own (university) students will typically ask, “Why should I study this?” I believe it is our duty to answer such questions, and I do not feel that this is being at all subservient. Education is a two-way process and over recent years has become much more of a partnership in learning rather than strictly didactic as it was less than half a century ago. Answers to the students’ question may range from highly practical ones — “Using these techniques we can reduce the price of petrol by one cent a litre,” to highly sophisticated ones — “This resolves a problem that has intrigued numbers of mathematicians for years.”

Given this situation, the *presentation* of the syllabus will change quite quickly, perhaps even from year to year. The *content* however, I would expect to change much more slowly. *Treatments* will also change. This last has been very noticeable in recent years with the advent, first of electronic calculators and more recently of computer packages, which do algebra and other parts of mathematics.

Presentation will depend on the political situation and the environment generally. Specific details will need to be worked out by the appropriate education department. In coming to conclusions here one must bear in mind that mathematics, like other disciplines, is something more enduring than our present whims.

Using technology. Although there are sometimes problems in using calculators (or computers) as noted above in §4, p. 180, when used for sensible calculations, calculators and computers can relieve an enormous amount of drudgery and save an enormous amount of time compared with doing calculations by hand. They also allow many variations with little effort. Here a nice example is the calculation of fractals, which became possible to visualize only through the use of computers. The use of computers can therefore assist in giving students experience of a wider range of mathematical *examples* than was possible a generation ago. However, it is important also to give the students an understanding of what they are doing as noted above in §4, p. 179.

The amount of material. With almost any syllabus there is a tendency to include more material at each revision. But there is also a tendency to retain everything from before. This too must be resisted.

Why can we delete items? There are many possible reasons. Most particularly the fact that technology has developed and can now be used to replace old calculations. But we need to be careful here.

In particular, given the intense and deep nature of mathematics it is even more important to make students aware of a mathematical approach and to try to set them on the path to mathematical maturity. In order to do this few subjects should be studied in depth rather than large numbers treated in a more superficial way. This is well summarized in a report from the US National Academy of Sciences:

> *High school courses for advanced study in mathematics and science should focus on helping students acquire in-depth understanding rather than the more superficial knowledge that comes from covering too much material too quickly, says a new report* [*Learning and Understanding: Improving Advanced Study of Mathematics and Science in U.S. High Schools*] *from the National Academies' National Research Council.* (Press release of the United States National Academy of Sciences, 14 February, 2002 accessed at: http://www4.nationalacademies.org/news.nsf/isbn/0309074401?OpenDocument)

This approach gives a guide to how to select what should be omitted from the syllabus. A balance will need to be struck between giving students generic skills and giving them enough experience (which must include giving them time to practice their skills) in a particular area. This is necessary to ensure both that students (a) understand the area and (b) are able to take the approach and apply it in a new area.

It is good to remember that Mathematics is a highly sophisticated *language*. Mathematics is hard in the same way that learning any language is hard. It requires practice. Therefore, and particularly in the areas where material is studied in depth, it is essential for the students to have time to practice what they have learnt/are learning.

Theory and practice

In providing a revised syllabus it is important to consider one's starting point. On the other hand, it is inappropriate to make huge changes in the syllabus. The side effects on the population of the whole school system would be too enormous – and too unpredictable (as they were some years ago with "New Mathematics").

In most syllabi that I have seen one considers the notion of *proof* throughout, but the notion and treatment of *structure* is less apparent.

People often think of graphs (In the sense of nodes joined by edges) and networks, when they think of structures today. This was not the case in much of the twentieth century. Such an example could easily be developed into the study of other structures. One, but only one, of the ultimate goals in the study of structures would be the understanding of the structure of computer programs – in particular the programs that will be *composed* by the students in the course of their calculations.

I believe that structure should have equal prominence with proof. One should not consider patterns only in number and a limited part of geometry, for example similar figures, and forget the many different sorts of structures that are amenable to mathematical treatment.

Structures are all around us and it is easy to

> *consider objects and activities from a mathematical perspective (for example, a netball game), detect and explain mathematical ideas in the features of natural or manufactured things (for example, the Myer Music Bowl in Melbourne or the Sydney Opera House), use mathematics to help explain a phenomenon or object in his or her environment, describe the mathematics in some visual representations of physical aspects of the world (for example metropolitan transport routes), investigate the meaning of numbers used to label and describe things met in daily life (for example, postcodes) and investigate the contribution made to mathematics by a particular individual or groups of people (for example, the Pythagoreans). Exposing students to this variety can surely only increase their awareness of how mathematics has been used to solve a broad range of problems, including key problems from historical contexts.* (From level 5 of the *Curriculum Standards Framework* (*II*), http://csf.vcaa.vic.edu.au/ma/lsma05.htm.)

Being able to understand structures is important for everyone, whether for reading a map, understanding the argument of a newspaper article or, in a more technical domain, making sense of mathematical equations or explanations that define a structure. Making students aware that these disparate examples are all examples of (mathematical) structures is important.[6] For this a beautiful, but rather obvious, example here is to be found in Federation Square in Melbourne, but other examples include the Sydney Opera House, the streets of Manhattan, the Judería in Córdoba, Spain, and the London underground.

The structure of the surfaces in Federation Square is the thing one notices on first acquaintance with the square. To understand the structure one needs to know about angles and triangles. To enjoy it to the full one also needs to know about congruence and symmetry.

A rather different example is provided by the Worldwide Web, which is familiar to all students. In terms of navigating the Web one has the same graph structure as following a street map. However there are many aspects of the structure of graphs that can be brought out – and having the language to talk about them would be very useful. For example, one could ask why some web-sites are easier to navigate than others, consider trees versus graphs, compare complexities (tree breadth and depth), and consider the interesting question of scaling (§ 6, p. 181) involved in collapsing a number of small pages into one big page, and conversely (see above, fn. 5, p. 181.)

However, in presenting examples such as the above it is very important that the students be helped to see the grand pattern and not to let them become bogged down in the minutiae of the examples or of routine repetition.

8 Some conclusions

Having determined what the context is in which mathematics is to be taught, and to whom, one needs to be aware of its human origins and also of its enormous potential for changing, and hopefully improving, the world.

[6] At this stage of schooling it would seem inappropriate to introduce the abstract idea of structure as such but nevertheless examples of (mathematical) structures abound and the language can be introduced. It is, of course, evident in some mathematics already, such as in the idea of similarity in geometry.

At the present time mathematics encompasses much more discrete mathematics than it ever did. This should be recognized and an informed choice be made between retaining what has traditionally been taught and what is needed now.

There is not time or space in school to retain everything presently in the syllabus and introduce more. That would also be unfair to both students and teachers. The best foundation is to treat a limited number of areas to a good depth and, at the same time, teach students how to learn mathematics so that they will later be able to venture into areas previously unknown to them.

Technology should be used where appropriate, but both its benefits and its limitations should be made clear. The rôles of both proof and computation should be clearly presented and permeate all the work.

Ideas of both proof and (mathematical) structure are fundamental and support each other. This should be properly recognized. However, without the idea of structure, the idea of proof has no foundation. Further the usefulness of the *language* of mathematics for describing the world, even without proof or calculation, should be established.

Finally a balance should be struck – and this is not easy to achieve – between, on the one hand, supporting, training and developing those students who will be our successors as teachers and professional mathematicians and, on the other, trying to give some idea, however minimal, of mathematics, its usefulness, power and beauty, to everyone.

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