Graph algebras: functional analysis with pictures

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About me

- My Dad's a mathematician, so there was never much hope for me...
- Undergraduate and PhD at the University of Newcastle — summer scholarships sucked me in.
- 6 months of my PhD at U. Iowa working with Paul Muhly.
- PhD on C^*-algebras of higher-rank graphs with Iain Raeburn.
- Very interesting examples arising from my thesis really kicked things off — collaboration with Pask, Raeburn and Rørdam.
- APD 2005–2007. Moved to Wollongong in 2007.
- Collaborations have been the key for me: 28 articles, 23 distinct co-authors. Used this to learn new areas and broaden: K-theory, noncommutative geometry, classification theory, groupoids, algebraic topology, coaction theory, product systems and representation theory, Dixmier-Douady theory...

C^*-algebras

- A C^*-algebra is an algebra of operators on Hilbert space.
- They arise in models for quantum statistical mechanics.
- The study of C^*-algebras has become a major focus since the mid 20$^{\text{th}}$ century.
- Connes, Conway, Jones and recently Tao have all been interested at one time or another.
- No systematic decomposition theorems as for groups.
- So tractable examples are key to the subject.

Graphs

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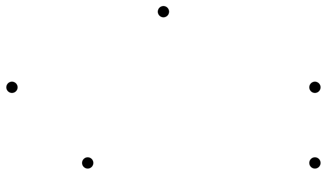

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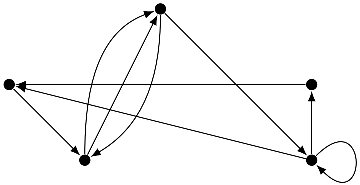

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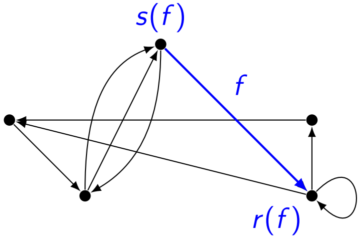
- The maps r and s indicate the directions of the arrows.



Graph C^*-algebras

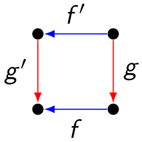
- Graph C^*-algebras "linearise" the dynamics of graphs.
- Represent on Hilbert space.
- Vertices $\leftrightarrow$ mutually orthogonal subspaces $\mathcal{H}_v$
- Edges $\leftrightarrow$ isometric linear maps $S_e : \mathcal{H}_{s(e)} \to \mathcal{H}_{r(e)}$.
- Require that each $\mathcal{H}_v = \bigoplus_{r(e)=v} S_e \mathcal{H}_{s(e)}$.
- So we have represented the dynamics of the graph as linear operators on Hilbert space.
- The norm-closed *-algebra generated by these operators is the graph C^*-algebra.

More on graph C^*-algebras

- Key theorems say that, under hypotheses, any two representations of a graph generate the same C^*-algebra.
- We can compute
  - The K-theory of $C^*(E)$;
  - The real and stable rank of $C^*(E)$;
  - The primitive ideal space of $C^*(E)$;
  - The trace simplex of $C^*(E)$;
  - approximate finite-dimensionality or pure infinite-ness.
- This tells us what examples can arise...
- ...and which ones can't.

Higher-rank graphs

- A higher-rank graph, or a k-graph, is like a k-dimensional version of a graph.
- So paths have a "shape" in $\mathbb{N}^k$ instead of a length in $\mathbb{N}$.



$$fg = g'f'$$

- Introduced by Kumjian and Pask in 2000 (just as I was starting my PhD).
- Can associate C^*-algebras to higher-rank graphs, but the theory is more complicated.

Higher-rank graph C^*-algebras

- Early on I worked on fundamental structure theory — ideal structure, uniqueness theorems.
- Examples arising from my thesis work showed that higher-rank graph C^*-algebras comprise a much larger class than graph C^*-algebras.
- Developing constructions and examples led to spin-offs in product systems, groupoids and Fell bundles and other areas.
- Question remains: exactly how broad is the class, and how can we use the results?
- Examples included "irrational rotation" algebras, but construction complicated and required classification theorems.
- Also, constructions somewhat ad hoc: no systematic approach.

Topological realisation

- Recent work involving Kaliszewski, Kumjian, Quigg, Pask, Whittaker takes a new approach.
- Suggested by earlier work of Pask-Quigg-Raeburn, and by connections to groupoids and topology.
- Idea: if paths in a higher-rank graph are “shaped” like rectangles, we should be able to “paste” honest rectangles into them to obtain a CW complex.
- This space has a fundamental group, a well-established covering theory, and notions of homology and cohomology.
- We are developing combinatorial notions of all of these, and proving that they agree.
- Investigating links to invariants of C^*-algebras.

Cohomology and twisted C^*-algebras

- In cohomology, a $\mathbb{T}$-valued 2-cocycle is a map $c : \{(\mu, \nu) : \mu\nu \text{ is a path}\} \to \mathbb{T}$ satisfying $c(\lambda, \mu)c(\lambda\mu, \nu) = c(\lambda, \mu\nu)c(\mu, \nu)$.
- The relations for the higher-rank graph C^*-algebra include $s_\mu s_\nu = s_{\mu\nu}$ when μ, ν are composable.
- The cocycle identity is precisely what we need to get an associative multiplication from $s_\mu s_\nu = c(\mu, \nu)s_{\mu\nu}$.
- The C^*-algebra only depends on the cohomology class of c.
- As elementary examples we obtain all irrational rotation algebras, all noncommutative tori, and a number of other examples that previously looked "sporadic."
- New systematic approach to these examples plus huge classes of related examples.
- Very exciting because there is so much to do.