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INDEPENDENT NON-IDENTICAL FIVE-PARAMETER GAMMA-WEIBULL VARIATES AND THEIR SUMS

ROY B. LEIPNIK[1] and CHARLES E. M. PEARCE[2]

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Abstract

Gamma-Weibull variates with five parameters are defined by multiplication of gamma and Weibull densities and renormalising. Sums of independent such variates are distributed as combinations of products of gammas and confluent hypergeometric functions and are explicitly determined. Sums of independent non-identical Weibulls arise as a special case. These variates can be used to model moderately extreme scenarios between gamma and Weibull that occur in many natural applications. All results are exact.

1. Introduction

In the first half of last century the foundations were laid for extreme value theory. As noted in a short and elegant overview article of Epstein [7], this occurred in the work of two non-interacting groups. Mathematicians, statisticians and probabilists (in particular in [5, 9, 10, 14, 25]) studied the asymptotic behaviour of sample extremes. For a later consolidated account of this work see [3, 13, 17, 20].

The second group were concerned with the failure of materials. The seminal paper was that of Griffith [16], who promulgated a "weakest link" theory. This was built on in [4, 8, 11, 12, 18, 22, 27] and [26]. A good collection of studies on applications is [6], the proceedings of a symposium in memory of W. Weibull. These centre on the Weibull distribution, which has cumulative distribution function

$$F(x) = F(x; b, \beta) = \begin{cases} 0 & x \le 0, \\ 1 - \exp\left(-bx^{\beta}\right) & x > 0. \end{cases}$$

Here b and β are positive parameters. This was introduced by Fisher and Tippett [9] in 1928 and rediscovered by Weibull [27, 26] in 1939 in the context of the failure of

[1]Mathematics Department, UCSB, Ca 93106–3080, USA; e-mail: leipnik@math.ucsb.edu.
[2]School of Mathematics, The University of Adelaide, SA 5005, Australia; e-mail: cpearce@maths.adelaide.edu.au.

elastic bodies under repeated stress and is known as a minimum-type Weibull [19]. The Weibull distribution may be shifted to a general origin $a > 0$ without difficulty, as is done below.

While Weibull distributions are well-suited to describing failure of an extreme nature in engineering, they often do not provide a good description of natural failures, such as in a biomedical context. This leads to the idea of using combined distributions, where two probability densities are multiplied together and renormalised to give a composite probability density. In this article we examine the gamma-Weibull distribution formed in this way. An underlying modelling notion is that a gamma "front-end" softens the "extremism" of Weibull behaviour.

Combined densities of the two classes contain more parameters and provide more fitting possibilities for situations of gradual ageing and weakening. Sums of such variates can be used for systems which contain a number of loosely coupled subsystems.

The Weibull distribution resembles the double-exponential distribution of Gumbel [17], which was also independently discovered by Fisher and Tippett. The gamma-Weibull distribution is quite similar to gamma-logistics (Balakrishnan and Leung [1]) and gamma-Gumbels (Rossi *et al*. [23]). See also Beran *et al*. [2] and Rossi *et al*. [24].

The sum of independent Gumbels, relating to concurrent catastrophes, was shown by Leipnik [21] to be unexpectedly difficult and attention turned to sums of Weibulls. Somewhat later, he was asked about a range of moderate failures, such as diabetes and hypertension, found in biomedical contexts. Combined gammas and Weibulls and their sums seemed both plausible and analytically simpler than gamma-Gumbels. The fact that the sum distributions can be represented exactly by well-known (and computed) special functions makes this class of distributions technically valuable. Sums of independent Fréchet distributions and gamma-Fréchets are also worth considering and can be handled similarly.

In Section 2 we address the basic gamma-Weibull distribution. Section 3 considers the sum of two independent gamma-Weibulls and Section 4 the sum of three or more gamma-Weibulls. The results can be easily specialised to the sum of independent Weibulls. We examine this in Section 5.

2. The gamma-Weibull distribution

If $x' = x - a$, then the density function $f(x)$ of a gamma-Weibull distribution $GW(b, \beta, \gamma, \mu, a)$ is defined by

$$\begin{aligned} f(x') &= K(x')^{\gamma-1} e^{-\mu x'} (x')^{\beta-1} e^{-bx'^{\beta}} \\ &= K(x')^{\gamma'-1} e^{-\mu x'} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(-bx'^{\beta}\right)^n, \end{aligned} \tag{2.1}$$

where $\gamma' = \gamma + \beta - 1$. This is a convergent expansion in gamma distributions for all $x' \geq 0$ and positive β, γ, b, μ. The constant K is needed for normalisation.

Since for each $n \geq 0$ the characteristic function for $e^{-\mu x}(x')^{\gamma'+n\beta-1}$ is

$$e^{iat}\Gamma(\gamma' + n\beta)\,(\mu - it)^{-(\gamma'+n\beta)}$$

for $\gamma' + n\beta > 0$, we derive that $f(x')$ has characteristic function

$$\phi^-(t) = K^- e^{iat} \sum_{n=0}^{\infty} \frac{(-b)^n}{n!}\,\Gamma(n\beta + \gamma')\,(\mu - it)^{-(\gamma'+n\beta)}, \tag{2.2}$$

which converges for $\beta < 1$. Here

$$(K^-)^{-1} = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!}\,\Gamma(n\beta + \gamma')\,(\mu)^{-(\gamma'+n\beta)}. \tag{2.3}$$

The superscript $^-$ denotes $\beta < 1$. All moments for $\beta < 1$ can be obtained from (2.2) by differentiation.

Even though (2.1) converges for all $\beta > 0$, neither (2.2) nor (2.3) does. However if $\beta > 1$, indicated by the superscript $^+$,

$$\begin{aligned}
\phi^+(t) &= K^+ e^{iat} \int_0^{\infty} x^{\gamma'-1} e^{-\mu x} e^{ixt} e^{-bx^\beta}\,dx \\
&= K^+ e^{iat} \int_0^{\infty} x^{\gamma'-1} \sum_{n=0}^{\infty} \frac{it - \mu)^n}{n!}\, e^{-bx^\beta}\,dx \\
&= \frac{K^+ e^{iat}}{\beta} \int_0^{\infty} e^{-y} \sum_{n=0}^{\infty} \frac{it - \mu)^n}{n!} \left(\frac{y}{b}\right)^{(n+\gamma')/\beta - 1} \frac{dy}{b}.
\end{aligned}$$

In the last step we have made the substitution $bx^\beta = y$. Since

$$\int_0^{\infty} e^{-y} y^{(n+\gamma')/\beta - 1}\,dy = \Gamma\left(\frac{n + \gamma'}{\beta}\right),$$

this provides

$$\phi^+(t) = \frac{K^+ e^{iat}}{\beta} \sum_{n=0}^{\infty} b^{-(n+\gamma')/\beta}\,\Gamma\left(\frac{n+\gamma'}{\beta}\right) \frac{it - \mu)^n}{n!}, \tag{2.4}$$

which converges for $\beta > 1$. Thus

$$\left(K^+\right)^{-1} = \frac{1}{\beta} \sum_{n=0}^{\infty} b^{-(n+\gamma')/\beta}\,\Gamma\left(\frac{n+\gamma'}{\beta}\right) \frac{-\mu)^n}{n!},$$

which also converges for $\beta > 1$. All moments for $\beta > 1$ can be calculated from (2.4).

When $\beta = 1$, the Weibull distribution collapses to an exponential distribution. Hence in this case f gives merely a gamma distribution, which is not of interest in the context of moderate failures.

3. Sum of two independent gamma-Weibulls

Because of the disjunction between the cases $\beta < 1$ and $\beta > 1$, there are three cases $(-,-)$, $(+,-)$ and $(+,+)$ of sums of two gamma-Weibulls. In the first case, the characteristic function of the sum is

$$\phi^{--} = \sum_{n_1,n_2=0}^{\infty} \prod_{j=1}^{2} \left[K_j^- e^{ia_j t} \frac{(-b_j)^{n_j}}{n_j!} \Gamma(n_j\beta_j + \gamma_j'(\mu_j - it)^{-(\gamma_j'+n_j\beta_j)} \right].$$

The inverse product of the transform is not well known and involves confluent hypergeometric functions. The convolution form

$$\begin{aligned} f_{12}(y)^{--} &= (f_1 * f_2)(y) \\ &= \sum_{n_1,n_2} \left[\prod_{j=1}^{2} \left\{ K_j^- \frac{(-b_j)^{n_j}}{n_j!} \right\} \right] \mathop{\circledast}_{j=1}^{2} \left((x_j')^{\gamma_j'+n_j\beta_j-1} e^{-\mu_j x_j'} \right). \end{aligned} \tag{3.1}$$

Here $*$ and $\circledast$ indicate convolution, $\beta_1, \beta_2 < 1$ and $x_1' + x_2' = y - a_1 - a_2$.

We have

$$\begin{aligned} &\left[\mathop{\circledast}_{j=1}^{2} \left((x_j^{\alpha} e^{-\mu_j x_j} \right) \right](y) \\ &= \int_0^y x^{\alpha_1} e^{-\mu_1 x} (y-x)^{\alpha_2} e^{-\mu_2(y-x)} dx = e^{-\mu_2 y} \int_0^y x^{\alpha_1} (y-x)^{\alpha_2} e^{x(\mu_2\mu_1)} dx \\ &= e^{-\mu_2 y} B(\alpha_1+1, \alpha_2+1) y^{\alpha_1+\alpha_2+1} {}_1F_1(\alpha_1+1, \alpha_1+\alpha_2+2; y(\mu_2-\mu_1)) \\ &= e^{-\mu_1 y} B(\alpha_1+1, \alpha_2+1) y^{\alpha_1+\alpha_2+1} {}_1F_1(\alpha_2+1, \alpha_1+\alpha_2+2; y(\mu_1-\mu_2)). \end{aligned} \tag{3.2}$$

The last step follows from the relation ${}_1F_1(a, b; z) = e^z {}_1F_1(b-a, b; -z)$ (see [15, p. 365, 3.383.1; p. 1086, 9.212.1]).

These hold for any $\alpha_1, \alpha_2 > 0$. Hence if $\mu_1 = \mu_2$, the ${}_1F_1$ factor above is replaced by unity, leading to a convergent double sum of weighted gamma densities. Setting $\alpha_j = \gamma_j' + n\beta_j - 1$ $(j = 1, 2)$ and double summing the suitably weighted terms from (3.1) and (3.2) over n_1, n_2 yields the desired density convolution on replacing y by $y - a_1 - a_2$.

We now have a formula for the sum of two gamma-Weibulls as a double series of weighted gamma densities multiplied by confluent hypergeometric functions. For the $(+,-)$ and $(+,+)$ cases only the normalisation in (3.1) needs to be changed, to $K_1^+ K_2^-$ and $K_1^+ K_2^+$ respectively. As seen from (3.2), the rest is unchanged in form.

4. Sum of many gamma-Weibulls

To add three gamma-Weibulls, the convolution of a gamma-Weibull and a gamma-hypergeometric is required. This reduces to many terms of the form

$$\begin{aligned}
&\int_0^y (y-x)^{\gamma'-1} e^{-(y-x)\mu} x^{\delta-1} e^{-x\nu} {}_1F_1(\alpha,\gamma';ax)\,dx \\
&\quad = e^{-\mu y} \int_0^y (y-x)^{\gamma'-1} e^{-x\omega} {}_1F_1(\alpha,\gamma';ax)\,dx \\
&\quad = e^{-\mu y} \int_0^y (y-x)^{\gamma'-1} x^{\delta-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (ax)^n}{(\gamma')_n n!} e^{x\omega}\,dx \\
&\quad = e^{-\mu y} \sum_{n=0}^{\infty} \int_0^y (y-x)^{\gamma'-1} x^{\delta+n-1} \frac{(\alpha)_n a^n}{(\gamma')_n n!} e^{x\omega}\,dx \\
&\quad = e^{-\mu y} \sum_{n=0}^{\infty} \frac{(\alpha)_n a^n}{(\gamma')_n n!} B(\gamma',\delta+n) y^{\gamma'+\delta+n-1} {}_1F_1(\gamma',\delta+n;y\omega),
\end{aligned}$$

where $\omega := \mu - \nu$.

This is again a combination of gamma densities and confluent hypergeometric functions, yielding a double series of this type. A sum of N independent gamma-Weibull variables will have as density an N-fold infinite series of gamma-confluent hypergeometric terms, with normalisations that reflect the β status of the individual variables. If $\beta_j > 1$ for X_j $(j = 1, \ldots, N_1)$ and $\beta_j < 1$ for X_j $(j = N_1 + 1, \ldots, N)$, then the normalisation will have the form $K_1^+ K_2^+ \ldots K_{N_1}^+ K_{N_1+1}^- \ldots K_N^-$ but the variable terms will have the same dependence form in $\beta_1, \ldots, \beta_N$, as with the sum of two gamma-Weibulls.

5. Sum of independent Weibulls

For two ordinary Weibulls, $\gamma_1 = \gamma_2 = 1$ and $\mu_1 = \mu_2 = 0$ (formally) and from (3.1) with $a_1 = a_2 = 0$ we have

$$f_{12}^{\pm\pm} = K_1^{\pm} K_2^{\pm} \sum_{mn} \frac{(-1)^{m+n}}{m!n!} b_1^m b_2^n B((m+1)\beta_1, (n+1)\beta_2) y^{(m+1)\beta_1 + (n+1)\beta_2 - 1}. \quad (5.1)$$

Here the K_1^+, K_1^-, K_2^+, K_2^- are selected according to the status of β_1, β_2 as explained earlier. When $a_1 \neq 0$ or $a_2 \neq 0$, the y on the right-hand side of (5.1) is replaced by $y - a_1 - a_2$.

Curiously, this expression converges less rapidly than the sum of two gamma-Weibulls. This is consistent with the notion mentioned earlier that a gamma "front end" softens the "extremism" of the Weibull behaviour. Similar remarks apply to the sum of three or more independent Weibulls. Extensive tables exist for ${}_1F_1$ and software in the IMSL library is quite efficient.

References

[1] N. Balakrishnan and M. Y. Leung, "Order statistics from the type I generalized logistic distribution", *Comm. Stat. Simulation Comput.* **17** (1988) 25–50.

[2] M. Beran, J. R. M. Hosking and N. Arnell, "Comment on 'Two-component extreme value distribution for flood frequency analysis' by Fabio Rossi, Mauro Fiorentino and Pasquale Versace", *Water Resources Res.* **22** (1986) 263–266.

[3] H. A. David, *Order statistics*, 2nd ed. (Wiley, New York, 1981).

[4] N. Davidenkov, E. Shevandin and F. Wittman, "The influence of size on the brittle strength of steel", *J. Appl. Mech.* **14** (1947) A63–67.

[5] E. L. Dodd, "The greatest and least variate under general laws of least error", *Trans. Amer. Math. Soc.* **25** (1923) 525–539.

[6] S. Eggwertz and N. C. Lind (eds.), *Probabilistic methods in the mechanics of solids and structures* (Springer, Berlin, 1985).

[7] B. Epstein, "Applications of extreme value theory to problems of material behavior", in *Probabilistic methods in the mechanics of solids and structures* (eds. S. Eggwertz and N. C. Lind), (Springer, Berlin, 1985) 3–9.

[8] J. C. Fisher and J. H. Hollomon, "A statistical theory of fracture", *Trans. Amer. Inst. Mining Metall. Eng.* **171** (1947) 546–561.

[9] R. A. Fisher and L. H. C. Tippett, "Limiting forms of the distribution of the largest or smallest member of a sample", *Proc. Camb. Phil. Soc.* **24** (1928) 180–190.

[10] M. Fréchet, "Sur la loi de probabilité de l'écart maximum", *Ann. de la Soc. de Math. (Cracow)* **6** (1927) 92–116.

[11] J. I. Frenkel and T. A. Kontorova, "Statistical theory of brittle strength of real crystals", *Zh. Tekhnich. Fiz.* **11** (1941) 173–183, (Russian).

[12] A. M. Freudenthal, "The statistical aspect of fatigue of materials", *Proc. Roy. Soc. London A* **187** (1946) 416–429.

[13] J. Galambos, *The asymptotic theory of extreme order statistics* (Wiley, New York, 1958).

[14] B. Gnedenko, "Sur la distribution limite du terme maximum d'une série aléatoire", *Ann. Math.* **44** (1943) 423–453.

[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 5th ed. (Academic Press, San Diego, 1994).

[16] A. A. Griffith, "The phenomena of rupture and flow in solids", *Phil. Trans Roy. Soc. A* **221** (1920) 163–198.

[17] E. J. Gumbel, *Statistics of extremes* (Columbia Univ. Press, New York, 1958).

[18] T. A. Kontorova, "Statistical theory of strength", *Zh. Tekhnich. Fiz.* **10** (1940) 886–890, (Russian).

[19] S. Kotz and S. Nadarajah, *Extreme value distributions* (Imperial College Press, London, 2000).

[20] M. R. Leadbetter, G. Lindgren and H. Rootzén, *Extremes and related properties of random sequences and processes* (Springer, New York, 1982).

[21] R. B. Leipnik, "Concurrent catastrophes: the sum of two independent Gumbels", *C. R. Math. Rep. Acad. Sci. Canada* **20** (1998) 65–70.

[22] F. T. Peirce, “Tensile tests for cotton yarns v. ‘The weakest link’— theorems on the strength of long and composite specimens”, *J. Textile Inst.* **17** (1926) 355–368.

[23] F. Rossi, M. Fiorentino and P. Versace, “Two-component extreme value distribution for flood frequency analysis”, *Water Resources Res.* **20** (1984) 847–856.

[24] F. Rossi, M. Fiorentino and P. Versace, “Reply”, *Water Resources Res.* **22** (1986) 267–269.

[25] R. von Mises, “La distribution de la plus grande de n valeurs”, *Rev. Math. Union Interbalkanique* **1** (1936) 141–160.

[26] W. Weibull, “The phenomenon of rupture in solids”, *Ing. Vetenskaps Akad. Handl.* **153** (1939) .

[27] W. Weibull, “A statistical theory of the strength of materials”, *Ing. Vetenskaps Akad. Handl.* **151** (1939) .