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GROWTH OF FUNCTIONS IN *CERCLES DE REMPLISSAGE*

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Abstract

Suppose that f is meromorphic in the plane, and that there is a sequence $z_n \to \infty$ and a sequence of positive numbers $\epsilon_n \to 0$, such that $\epsilon_n |z_n| f^{\#}(z_n)/\log |z_n| \to \infty$. It is shown that if f is analytic and non-zero in the closed discs $\Delta_n = \{z : |z - z_n| \le \epsilon_n |z_n|\}$, $n = 1, 2, 3, \dots$, then, given any positive integer K, there are arbitrarily large values of n and there is a point z in Δ_n such that $|f(z)| > |z|^K$. Examples are given to show that the hypotheses cannot be relaxed.

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1. Introduction

For a meromorphic function f, discs of the form

$$\Delta(z_n, \epsilon_n |z_n|) = \{z : |z - z_n| < \epsilon_n |z_n|\},$$

where $z_n \to \infty$ and $\epsilon_n \to 0$, are called *cercles de remplissage* if f takes every extended complex value with at most two exceptions infinitely often in any infinite subcollection of them. Lehto [2] pointed out the close connection between the spherical derivative:

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

and *cercles de remplissage*: if, for a sequence $z_n \to \infty$, there exist positive numbers $\epsilon_n \to 0$ such that

$$\epsilon_n |z_n| f^{\#}(z_n) \to \infty,$$

then $\Delta(z_n, 2\epsilon_n|z_n|)$ is a sequence of *cercles de remplissage*; and conversely, if $\Delta(z_n, \epsilon_n|z_n|)$ is a sequence of *cercles de remplissage*, then for each n there is a point $z_n' \in \Delta(z_n, 2\epsilon_n|z_n|)$ such that

$$\epsilon_n|z_n'|\, f^{\#}(z_n') \to \infty.$$

This note is concerned with the way in which the growth of f in *cercles de remplissage* is related to the growth of $f^{\#}$. We will prove:

THEOREM 1. *Suppose that f is meromorphic in the plane, and that there is a sequence $z_n \to \infty$ and a sequence of positive numbers $\epsilon_n \to 0$, such that*

$$\epsilon_n|z_n|\, f^{\#}(z_n)/\log|z_n| \to \infty. \tag{1}$$

If f is analytic and non-zero in the closed discs

$$\Delta_n = \{z : |z - z_n| \le \epsilon_n|z_n|\}, \quad n = 1, 2, 3, \ldots,$$

then f grows transcendentally there, that is, given any positive integer K, there are arbitrarily large values of n and there is a point z in Δ_n such that

$$|f(z)| > |z|^K. \tag{2}$$

(Notice that, from Lehto's theorem, the discs Δ_n form a sequence of *cercles de remplissage*; in fact the radius could be reduced to $2\epsilon_n|z_n|/\log|z_n|$.) Theorem 1 cannot be significantly improved. As we will show, there is a function f, and a sequence $z_n \to \infty$, such that

$$0 < \lim_{n\to\infty} |z_n|\, f^{\#}(z_n)/\log|z_n| < \infty, \tag{3}$$

and such that f is non-zero in $\Delta(z_n, |z_n|/2)$ and satisfies $f(z) = O(z)$ in $\Delta(z_n, |z_n|/2)$. Thus (1) cannot be relaxed. Also, there is a function f, and there are sequences $z_n \to \infty$ and $\epsilon_n \to 0$, such that (1) just holds, while $f(z) = O(z)$ in $\Delta(z_n, \epsilon_n|z_n|)$. Every such z_n is a zero of f, and thus the hypothesis that f does not vanish in Δ_n cannot be omitted from Theorem 1.

Two remarks are in order. First of all, in [1], the authors use the fact that all transcendental entire functions satisfy (1) to prove the existence of *cercles de remplissage* in which (2) holds. One need not assume that the function is non-zero in these *cercles*. Theorem 1 together with the second example mentioned above show that the situation is quite different for meromorphic functions.

Secondly, Theorem 1 is connected with the existence of Hayman directions. A direction $\theta \in [0, 2\pi]$ is said to be a *Hayman direction* for a meromorphic function

f if, given $\epsilon > 0$, either f takes all complex values infinitely often in the region $D = \{z : |\arg z - \theta| < \epsilon\}$ or else all its derivatives take all complex values, except possibly zero, infinitely often there. The authors have shown [1, Theorem 2] that every transcendental entire function has a Hayman direction. The proof depends on the fact that every transcendental entire function has a sequence of *cercles de remplissage* in which it grows transcendentally [1, Theorem 1], and it is easily seen that, in view of Theorem 1 of the present paper, the same argument can be used to prove the following theorem:

THEOREM 2. *Every meromorphic function satisfying* (1) *has a Hayman direction.*

This complements a result of Yang Lo [5], who showed that a meromorphic function f has a Hayman direction if

$$\limsup_{r\to\infty} T(r, f)/(\log r)^3 = \infty. \tag{4}$$

The two statements appear to be independent.

2. Proof of Theorem 1

If Theorem 1 is false then there is a meromorphic function f, analytic and non-zero in the union of the discs Δ_n, and a positive number K, such that

$$|f(z)| \le |z|^K, \quad z \in \Delta_n, \tag{5}$$

for all large n. For $z \in \Delta_n$, write $f(z) = e^{p_n(z)}$, where p_n is analytic in Δ_n, so that

$$f^{\#}(z) = |p_n'(z)| \frac{e^{u_n(z)}}{1 + e^{2u_n(z)}}, \tag{6}$$

where $u_n = \text{Re}\, p_n$. Since $0 \le t/(1+t^2) \le 1/2$ if $t \ge 0$, $f^{\#}(z) \le (1/2)|p_n'(z)|$, and therefore

$$|p_n'(z_n)| \ge 2M_n \log|z_n|/|z_n|, \tag{7}$$

where

$$M_n = |z_n| f^{\#}(z_n)/\log|z_n|.$$

Writing $p_n(z) = \sum_{j=0}^{\infty} a_j(n)(z - z_n)^j$ and defining $A_n(t) = \max_{|z-z_n|=t} u_n(z)$, we have, from (5), $A_n(t) \le K \log|z_n|$, for $0 \le t \le \epsilon_n|z_n|$. Further [4, page 86],

$$|a_j(n)|t^j \le 4\max\{A_n(t), 0\} - 2u_n(z_n),$$

for all j. With $j = 1$ and $t = \epsilon_n|z_n|$, we obtain

$$\begin{aligned} |p_n'(z_n)| &\le (4\max\{A_n(\epsilon_n|z_n|), 0\} - 2u_n(z_n))/(\epsilon_n|z_n|) \\ &\le (4K\log|z_n| - 2u_n(z_n))/(\epsilon_n|z_n|), \end{aligned} \tag{8}$$

and therefore, in view of (7),

$$\epsilon_n M_n \log|z_n| \le 2K\log|z_n| - u_n(z_n).$$

Since $\epsilon_n M_n \to \infty$, from (1), we deduce that $u_n(z_n) \le -(1+o(1))\epsilon_n M_n \log|z_n|$, and further, from (8),

$$|p_n'(z_n)| \le (2 + o(1))|u_n(z_n)|/(\epsilon_n|z_n|).$$

Returning to (6), we obtain, with $z = z_n$,

$$\epsilon_n|z_n|\, f^{\#}(z_n) \le (2 + o(1))|u_n(z_n)|\, e^{u_n(z_n)} \to 0.$$

But $\epsilon_n|z_n|\, f^{\#}(z_n) = \epsilon_n M_n \log|z_n| \to \infty$, a contradiction, which proves the theorem. □

3. Two examples

The first example is

$$f(z) = \prod_{n=1}^{\infty} \frac{z + e^{\sqrt{n}}}{z - e^{\sqrt{n}}}. \tag{9}$$

Rossi [3] has shown that $f(z) = O(z)$ in any small sector about the imaginary axis, and f is evidently non-zero there. We will show that

$$\lim_{t\to+\infty} t f^{\#}(\pm it)/\log t = \pi/2. \tag{10}$$

Differentiating $\log f$ we have, with $z = \pm it$,

$$\frac{f'(z)}{f(z)} = \sum_{1}^{\infty} \frac{2e^{\sqrt{n}}}{t^2 + e^{2\sqrt{n}}}. \tag{11}$$

Fix N such that $e^{\sqrt{N}} \le t < e^{\sqrt{N+1}}$. Since $X/(T^2 + X^2)$ increases for $0 \le X \le T$ and decreases after that,

$$\int_0^{N-1} \frac{e^{\sqrt{x}}}{t^2 + e^{2\sqrt{x}}} dx \le \sum_{1}^{N-1} \frac{e^{\sqrt{n}}}{t^2 + e^{2\sqrt{n}}} \le \int_1^{N} \frac{e^{\sqrt{x}}}{t^2 + e^{2\sqrt{x}}}\, dx,$$

and

$$\int_{N+3}^{\infty} \frac{e^{\sqrt{x}}}{t^2+e^{2\sqrt{x}}}dx \le \sum_{N+2}^{\infty} \frac{e^{\sqrt{n}}}{t^2+e^{2\sqrt{n}}} \le \int_{N+1}^{\infty} \frac{e^{\sqrt{x}}}{t^2+e^{2\sqrt{x}}}\, dx.$$

Using simple estimates on terms of the form $e^{\sqrt{w}}/(t^2+e^{\sqrt{w}})$, it follows that

$$\sum_{1}^{\infty} \frac{e^{\sqrt{n}}}{t^2+e^{2\sqrt{n}}} = \int_{0}^{\infty} \frac{e^{\sqrt{x}}}{t^2+e^{2\sqrt{x}}}\, dx + O(t^{-1}).$$

After a change of variable the integral on the right hand side is

$$\frac{2}{t}\int_{1/t}^{\infty} \frac{\log u + \log t}{1+u^2}\, du,$$

and since

$$\int_{0}^{\infty} \frac{\log u}{1+u^2}\, du = 0 \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{1+u^2}\, du = \frac{1}{2}\pi,$$

we obtain, from (11),

$$\frac{f'(z)}{f(z)} = (\pi + o(1))\frac{\log t}{t}.$$

And (10) follows from this since, for $z = \pm it$, $|f(z)| = 1$.

The second example is

(12) $$f(z) = \prod_{n=1}^{\infty} \frac{1-z/a_n}{1-z/b_n},$$

where $a_n = e^{3^n/\psi_n}$, $b_n = 3a_n$, and ψ_n is a positive increasing sequence that tends slowly to infinity, and is such that, for all large N,

(13) $$3^n/\psi_n \le 3^N/(2\psi_N), \quad n < N.$$

Let $\epsilon_n = 1/\sqrt{\psi_n}$. Using the fact that $a_n < \sqrt{a}_N$ if $n < N$ and $a_n > \sqrt{a}_N$ if $n > N$, it follows from (13) that, for $|z - a_N| < \epsilon_N a_N$,

$$(1-z/a_n)/(1-z/b_n) = (1+O(e^{-3^N/2\psi_N}))b_n/a_n,$$

for $n < N$, and

$$(1-z/a_n)/(1-z/b_n) = 1+O(e^{-3^n/2\psi_n}),$$

for $n > N$. Thus, if $|z - a_N| < \epsilon_N a_N$,

(14) $$f(z) = (1 + o(1))3^{N-1}(1 - z/a_N)/(1 - z/b_N) = (1 + o(1))3^N(1 - z/a_N).$$

Writing $F(z) = f(z)/(1 - z/a_N)$, we obtain $f'(a_N) = -F(a_N)/a_N$, and so, by (14),

$$f'(a_N) = -(1 + o(1))3^N/a_N.$$

We conclude that

$$\epsilon_n a_N f^{\#}(a_N)/\log a_N = (1 + o(1))3^N \epsilon_n/\log a_N = \sqrt{\psi_N},$$

so that (1) holds with $z_n = a_n$. Moreover, from (14), if $|z - a_N| < \epsilon_N a_N$, then

$$|f(z)| \le (1 + o(1))3^N \epsilon_N = (1 + o(1))3^N/\sqrt{\psi_N} \le (1 + o(1))\sqrt{\psi_N} \log a_N.$$

Thus, with $\psi_n = n$ say, which satisfies (13) for all $N \ge 3$, f does not grow transcendentally in the discs $\Delta(a_n, \epsilon_n a_n)$.

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