Let $G$ be a locally compact Vilenkin group with dual group $\Gamma$. We prove Littlewood-Paley type inequalities corresponding to arbitrary coset decompositions of $\Gamma$. These inequalities are then applied to obtain new $L^p(G)$ multiplier theorems. The sharpness of some of these results is also discussed.


1. Introduction

Given a sequence $\{g_n\}$ of Fourier multipliers for $L^p(\mathbb{R})$, $1 < p < \infty$, let $g := \sum_{n=-\infty}^{\infty} g_n \chi_n$, where $\chi_n$ denotes the characteristic function of the dyadic interval $[2^n, 2^{n+1}]$ in $\mathbb{R}$. In an earlier paper [OQ] we proved that if the sequence $\{g_n\}$ belongs to a certain mixed-norm space, then $g$ is also an $L^p(\mathbb{R})$ multiplier. A similar result was established for Fourier multipliers for $L^p(G)$-spaces, where $G$ is a locally compact Vilenkin group. In that case we considered the decomposition of $\Gamma$, the dual group of $G$, into sets that are comparable to the dyadic intervals in $\mathbb{R}$.

In this paper we consider essentially the same problem for decompositions of $\Gamma$ into a union of arbitrary disjoint cosets of subgroups of $\Gamma$. The proof of the resulting multiplier theorem, Theorem 5, depends on a one-sided extension of the Littlewood-Paley inequality in the context of Vilenkin groups. This generalizes a similar result of Rubio de Francia for functions in $L^p(\mathbb{R})$, $2 \leq p < \infty$. We also prove another one-sided Littlewood-Paley-type inequality for functions in $L^p(G)$, $1 < p < 2$. This inequality is then used to obtain an additional multiplier theorem, Theorem 6. Finally, we discuss the sharpness of some of our results, see Theorems 7, 8 and 9.
2. Definitions and notation

Throughout this paper \( G \) will denote a locally compact Vilenkin group, that is to say, \( G \) is a locally compact Abelian topological group containing a strictly decreasing sequence of open compact subgroups \( (G_n)_{n=1}^{\infty} \) such that \( \bigcup_{n=1}^{\infty} G_n = G \) and \( \bigcap_{n=1}^{\infty} G_n = \{0\} \). In [EG, Section 4.1.4] such groups are called groups with a suitable family of compact open subgroups \( (G_n)_{n=1}^{\infty} \). Clearly, such groups are totally disconnected. Examples of locally compact Vilenkin groups are the \( p \)-adic numbers and, more generally, the additive group of a local field, see [EG] or [Ta] for further details.

Let \( \Gamma \) denote the dual group of \( G \), and for each \( n \in \mathbb{Z} \), let \( \Gamma_n \) denote the annihilator of \( G_n \), that is,

\[ \Gamma_n = \{ \gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n \}. \]

Then we have \( \bigcup_{n=1}^{\infty} \Gamma_n = \Gamma \), \( \bigcap_{n=1}^{\infty} \Gamma_n = \{1\} \) and order \( (\Gamma_{n+1}/\Gamma_n) \) = order \( (G_n/G_{n+1}) \) for all \( n \in \mathbb{Z} \).

We choose Haar measures \( \mu \) on \( G \) and \( \lambda \) on \( \Gamma \) so that \( \mu(G_0) = \lambda(\Gamma_0) = 1 \). Then \( \mu(G_n) = (\lambda(\Gamma_n))^{-1} \) for all \( n \in \mathbb{Z} \); we set \( m_n := \lambda(\Gamma_n) \).

For \( p \) with \( 1 \leq p \leq \infty \) we shall denote its conjugate by \( p' \); thus \( 1/p + 1/p' = 1 \). For an arbitrary set \( E \) we denote its characteristic function by \( \chi_E \). The symbols \( \wedge \) and \( \vee \) will be used to denote the Fourier and inverse Fourier transform, respectively. It is easy to see that for each \( n \in \mathbb{Z} \) we have

\[ (\chi_{\Gamma_n})^\vee = (\mu(G_n))^{-1} \chi_{G_n} := \Delta_n. \]

For a definition of the spaces of test functions and distributions on \( G \) and \( \Gamma \), see [Ta]; these spaces will be denoted by \( \mathcal{S}(G) \), \( \mathcal{S}'(G) \), \( \mathcal{S}(\Gamma) \) and \( \mathcal{S}'(\Gamma) \). We can also extend the Fourier and inverse Fourier transform to \( \mathcal{S}'(G) \) and \( \mathcal{S}'(\Gamma) \) in the standard way and the usual properties hold, see [Ta] for details.

Let \( f \) be a locally integrable function on \( G \). The function \( M_2 f \) is defined on \( G \) by

\[ M_2 f(x) := \sup_{k \in \mathbb{Z}} \left\{ \frac{1}{\mu(x + G_k)} \int_{x+G_k} |f(y)|^2 d\mu(y) \right\}^{1/2}. \]

Thus \( M_2 f = \{ M(|f|^2) \}^{1/2} \), where \( M \) is the Hardy-Littlewood maximal operator on \( G \).

The sharp function \( f^\# \) is defined on \( G \) by

\[ f^\#(x) := \sup_{n \in \mathbb{Z}} \left\{ \frac{1}{\mu(x + G_n)} \int_{x+G_n} |f(y) - f_{x+G_n}| d\mu(y) \right\}, \]

where

\[ f_{x+G_n} = \frac{1}{\mu(x + G_n)} \int_{x+G_n} f(y) d\mu(y). \]
For $1 \leq p \leq \infty$ let $L^p(G)$ be the space of all $p$-th integrable functions on $G$, with obvious modification for $p = \infty$. For a measurable function $f$ on $G$ we set

$$\sigma(f, y) = \mu\{x \in G : \ |f(x)| > y\}, \quad y > 0,$$

and

$$f^*(t) = \inf\{y > 0 : \ \sigma(f, y) \leq t\}, \quad t > 0.$$

For $1 \leq p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L^{p,q}(G)$ is the collection of all measurable functions $f$ on $G$ such that $\|f\|_{L^{p,q}(G)} < \infty$, where

$$\|f\|_{L^{p,q}(G)}^* = \begin{cases} \frac{q}{p} \int_0^{\infty} (t^{1/p} f^*(t))^{q/d} \frac{dt}{t} \quad & \text{if } 1 \leq p < \infty, \quad 1 \leq q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) \quad & \text{if } 1 \leq p < \infty, \quad q = \infty. \end{cases}$$

Next, the function $f^{**}$ is defined on $\mathbb{R}^+$ by

$$f^{**}(t) = \sup_{t \leq \mu(E)} \left\{ \frac{1}{\mu(E)} \int_E |f(x)|^{1/2} d\mu(x) \right\}^2.$$

We denote $\|f^{**}\|_{L^{p,q}(\mathbb{R}^+)}$ by $\|f\|_{L^{p,q}(G)}$. It is easy to see that $(f^{**})^* = f^{**}$ and $f^*(t) \leq f^{**}(t) \leq (f^{**})^*(t)$ for all $t > 0$. Hence we have

$$\|f\|_{L^{p,q}(G)} \leq \|f\|_{L^{p,q}(G)} \leq \|f^*\|_{L^{p,q}(\mathbb{R}^+)}.$$

By Hardy’s inequality we also have

$$\|f^*\|_{L^{p,q}(\mathbb{R}^+)} \leq C \|f\|_{L^{p,q}(G)}.$$

We note that $L^{p,q}(G) \subseteq L^{p,s}(G)$ if $q \leq s$. We equip $L^{p,q}(G)$ with either $\| \cdot \|_{L^{p,q}(G)}^*$ or $\| \cdot \|_{L^{p,q}(G)}$ to define its topology. We observe that $L^{p,p}(G) = L^p(G)$ and we simply denote $\| \cdot \|_{L^{p,q}(G)}$ by $\| \cdot \|_{p,q}$ and $\| \cdot \|_{p,p}$ by $\| \cdot \|_p$ if there is no confusion likely. The same notational simplifications also apply to $\| \cdot \|_{L^{p,q}(\mathbb{R}^+)}$.

Let $\phi \in L^\infty(\Gamma)$ and define $T_\phi$ on $\mathcal{S}(G)$ by $(T_\phi f)^\wedge = \hat{\phi} \hat{f}$, $f \in \mathcal{S}(G)$. The function $\phi$ is said to be a multiplier from $L^{p,q}(G)$ into $L^{r,s}(G)$ if there exists a positive constant $C$ so that for all $f \in \mathcal{S}(G)$ we have

$$\|T_\phi f\|_{r,s} \leq C \|f\|_{p,q},$$

where $1 \leq p, r < \infty$, $1 \leq q, s \leq \infty$. We say that $\phi$ is a multiplier of weak type $(p, p)$ if it is a multiplier from $L^p(G)$ to $L^{p,\infty}(G)$. The collection of all multipliers from $L^p(G)$ into $L^p(G)$ is denoted by $\mathcal{M}(L^p(G))$ and the corresponding multiplier norm is denoted by $\| \cdot \|\mathcal{M}(L^p)$. 
3. A Littlewood-Paley inequality for arbitrary coset decompositions of \( \Gamma \); the case \( 2 \leq p < \infty \)

Let \( \{J_k\}_{k=0}^{\infty} \) be a sequence of mutually disjoint intervals of \( \mathbb{R} \). For \( f \in L^1(\mathbb{R}) \) and \( 1 < r < \infty \) define the function \( \tilde{\Delta}_r f \) on \( \mathbb{R} \) by

\[
\tilde{\Delta}_r f := \left( \sum_{k=0}^{\infty} |S_{I_k} f|^r \right)^{1/r},
\]

where

\[
(S_{I_k} f)^\wedge(\xi) := \chi_{I_k}(\xi) \hat{f}(\xi).
\]

The following result was proved by Rubio de Francia in [R, Theorem 1.2].

**THEOREM R.** Let \( 2 \leq p < \infty \). There exists a constant \( C_p \) such that

\[
\| \tilde{\Delta}_2 f \|_p \leq C_p \| f \|_p, \quad f \in L^p(\mathbb{R}).
\]

In [Sj] Sjölin gave a different proof of Theorem R. In this section we use Sjölin’s method to obtain an analogue of Theorem R on locally compact Vilenkin groups \( G \).

**THEOREM 1.** Let \( 2 \leq p < \infty \) and let \( \{\Lambda_k\}_{k=0}^{\infty} := \{\gamma_k + \Gamma_{n_k}\}_{k=0}^{\infty} \) be a decomposition of \( \Gamma \) into mutually disjoint cosets of various subgroups of \( \Gamma \). For \( f \in \mathcal{S}(G) \) define the function \( \Delta f \) on \( G \) by

\[
\Delta f := \left( \sum_{k=0}^{\infty} |S_{\Lambda_k} f|^2 \right)^{1/2},
\]

where

\[
(S_{\Lambda_k} f)^\wedge(\gamma) := \chi_{\Lambda_k}(\gamma) \hat{f}(\gamma).
\]

Then

\[
\| \Delta f \|_p \leq C_p \| f \|_p
\]

and this inequality can be extended to all \( f \in L^p(G) \).

**PROOF.** It follows immediately from Plancherel’s equality that

\[
\| \Delta f \|_2 = \| f \|_2.
\]
Thus we may assume that $2 < p < \infty$. For each $k \geq 0$ we define $\psi_k : G \to \mathbb{C}$ by $\psi_k(x) = \gamma_k(x) \Delta_n(x)$, so that $(\psi_k)^\wedge(\gamma) = \chi_{\Gamma_{n_k}}(\gamma - \gamma_k) = \chi_{\Lambda_k}(\gamma)$. Thus for $f \in \mathcal{S}(G)$ we have

$$\Delta f(x) = \left\{ \sum_{k=0}^{\infty} |\psi_k \ast f(x)|^2 \right\}^{1/2}.$$ 

The theorem will follow from the following string of inequalities as in Rubio de Francia [R, p. 5]:

$$\|\Delta f\|_p \leq C \|\Delta f\|_p^\# \leq C \|M_2 f\|_p \leq C \|f\|_p.$$

It is clear that the last inequality holds as long as $2 < p < \infty$ and we only have to justify the second inequality the proof of which will be given in Lemma 1 below.

**Lemma 1.** Let $f \in \mathcal{S}(G)$. Then $(\Delta f)^\#(x) \leq C M_2 f(x)$ for all $x \in G$.

**Proof.** Take any $x_0 \in G$ and let $I_0 := x_0 + G_{k_0}$ be a coset containing $x_0$. Decompose $f$ into

$$f = f \chi_{I_0} + f \chi_{G \setminus I_0} := g + h.$$ 

Let

$$a := \left( \sum_{k \in S_0} |\psi_k \ast h(x_0)|^2 \right)^{1/2},$$

where $S_0 = \{ k : n_k \leq k_0 \}$; that is to say, we sum over those values of $k$ for which the corresponding function $\psi_k$ has the property:

$$G_{k_0} \subset G_{n_k} = \text{supp } (\psi_k).$$

For every $x \in G$ we have

$$|\Delta f(x) - a| \leq |\Delta f(x) - \Delta h(x)| + |\Delta h(x) - a|. \hspace{1cm} (\dagger)$$

We analyze each of the two terms in $(\dagger)$. By the $\ell^2$-triangle inequality we have

$$\Delta f(x) = \left( \sum_k \left| \psi_k \ast g + \psi_k \ast h \right|^2(x) \right)^{1/2} \leq \left( \sum_k \left| \psi_k \ast g(x) \right|^2 \right)^{1/2} + \left( \sum_k \left| \psi_k \ast h(x) \right|^2 \right)^{1/2} = \Delta g(x) + \Delta h(x),$$
that is,

$$\Delta f(x) - \Delta h(x) \leq \Delta g(x).$$

Similarly,

$$\Delta h(x) = \Delta (f-g)(x) \leq \Delta f(x) + \Delta g(x)$$

so that

$$\Delta h(x) - \Delta f(x) \leq \Delta g(x).$$

Therefore,

$$|\Delta f(x) - \Delta h(x)| \leq \Delta g(x).$$

For the second term in (†) we have

$$|\Delta h(x) - a| = \left| \left( \sum_{k} |\psi_k * h(x)|^2 \right)^{1/2} - \left( \sum_{k \in S_0} |\psi_k * h(x_0)|^2 \right)^{1/2} \right|$$

$$= \left| \left( \sum_{k} |\psi_k * h(x) \gamma_k(x)|^2 \right)^{1/2} - \left( \sum_{k \in S_0} |\psi_k * h(x_0) \gamma_k(x_0)|^2 \right)^{1/2} \right|$$

$$\leq \left( \sum_{k \notin S_0} |\psi_k * h(x)|^2 \right)^{1/2} + \left( \sum_{k \in S_0} |F_k(x)|^2 \right)^{1/2},$$

where

$$F_k(x) := \psi_k * h(x) \gamma_k(x) - \psi_k * h(x_0) \gamma_k(x_0)$$

$$= \int_{G} \psi_k(x-y)h(y)\gamma_k(x)dy - \int_{G} \psi_k(x_0-y)h(y)\gamma_k(x_0)dy$$

$$= \int_{G} [\Delta n_k(x-y) - \Delta n_k(x_0-y)]\gamma_k(y)h(y)dy.$$

Thus we see that

$$|\Delta f(x) - a| \leq \Delta g(x) + \left( \sum_{k \notin S_0} |\psi_k * h(x)|^2 \right)^{1/2} + \left( \sum_{k \in S_0} |F_k(x)|^2 \right)^{1/2}$$

$$:= A_1(x) + A_2(x) + A_3(x).$$
We now consider in turn \( \frac{1}{\mu(I_0)} \int_{I_0} A_i(x) dx \), \( i = 1, 2, 3 \).

We have
\[
\frac{1}{\mu(I_0)} \int_{I_0} A_1(x) dx = m_k \int_{I_0} \Delta g(x) dx \\
\leq m_k \left( \int_{I_0} |\Delta g(x)|^2 dx \right)^{1/2} \left( \int_{I_0} dx \right)^{1/2} \\
\leq (m_k)^{1/2} \left( \int_{\Gamma} \sum_k |\psi_k * g(x)|^2 dx \right)^{1/2} \\
\leq (m_k)^{1/2} \left( \sum_k \int_{\Gamma} (|\psi_k * g|^2)^{1/2} \right)^{1/2} \\
= (m_k)^{1/2} \left( \int_{I_0} |g(x)|^2 dx \right)^{1/2} \\
= \left( \frac{1}{\mu(I_0)} \int_{I_0} |f(x)|^2 dx \right)^{1/2} \\
\leq CM_2 f(x_0).
\]

To find an estimate for
\[
\frac{1}{\mu(I_0)} \int_{I_0} A_2(x) dx = \frac{1}{\mu(I_0)} \int_{I_0} \left( \sum_{k \notin S_0} |\psi_k * h(x)|^2 \right)^{1/2} dx
\]
we observe that for \( x \in I_0 \) and \( k \notin S_0 \) we have
\[
|\psi_k * h(x)| = |\psi_k * h(x) \gamma_k(x)| = \left| \int_{G} \psi_k(x - y) h(y) \gamma_k(x) dy \right| \\
= \left| \int_{G} \Delta_{n_k}(x - y) \gamma_k(y) h(y) dy \right| \\
= \left| \int_{G \setminus I_0} \Delta_{n_k}(x - y) \gamma_k(y) f(y) dy \right|.
\]

For \( x \in I_0 = x_0 + G_{k_0} \) and \( y \notin I_0 \) we have \( x - y \notin G_{k_0} \). Also, \( k \notin S_0 \) implies that \( G_{n_k} \subset G_{k_0} \). Thus \( x - y \notin G_{n_k} \) and, hence, \( \Delta_{n_k}(x - y) = 0 \). That is,
\[
\frac{1}{\mu(I_0)} \int_{I_0} A_2(x) dx = 0.
\]
To find an estimate for

\[ \frac{1}{\mu(I_0)} \int_{I_0} A_3(x) \, dx = \frac{1}{\mu(I_0)} \int_{I_0} \left( \sum_{k \in S_0} |F_k(x)|^2 \right)^{1/2} \, dx \]

we observe that

(i) if \( x \in I_0 \) and \( k \in S_0 \) and \( y \in x_0 + G_{n_k} \) then we have \( x - y \in G_{n_k} \) and \( x_0 - y \in G_{n_k} \), so that

\[ \Delta_{n_k}(x - y) - \Delta_{n_k}(x_0 - y) = m_{n_k} - m_{n_k} = 0. \]

(ii) if \( x \in I_0 \) and \( k \in S_0 \) and \( y \notin x_0 + G_{n_k} \) then \( x - y \notin G_{n_k} \) and \( x_0 - y \notin G_{n_k} \), so that

\[ \Delta_{n_k}(x - y) - \Delta_{n_k}(x_0 - y) = 0. \]

We see that for \( x \in I_0 \) and \( k \in S_0 \) we have \( F_k(x) = 0 \), so that

\[ \frac{1}{\mu(I_0)} \int_{I_0} A_3(x) \, dx = 0. \]

Thus we may conclude that

\[ \frac{1}{\mu(I_0)} \int_{I_0} |\Delta f(x) - a| \, dx \leq C M_2 f(x_0), \]

so that

\[ (\Delta f)(x_0) \leq C M_2 f(x_0). \]

This completes the proof of the Lemma.

4. A Littlewood-Paley-type inequality for arbitrary coset decompositions of \( \Gamma \); the case \( 1 < p < 2 \)

For the case \( 1 < p < 2 \), Rubio de Francia conjectured that for each \( f \in L^p(\mathbb{R}) \) we have

\[ \| \tilde{\Delta}_p f \|_p \leq C_p \| f \|_p. \]

In this section we shall prove an inequality that is related to but weaker than the inequality in Rubio de Francia’s conjecture.
THEOREM 2. Let \( 1 < p < 2 \) and let \( \{ \Lambda_k \}_{k=0}^{\infty} := \{ \gamma_k + \Gamma_{n_k} \}_{k=0}^{\infty} \) be a decomposition of \( \Gamma \) into mutually disjoint cosets of various subgroups of \( \Gamma \). If \( T \) is the operator defined for a simple function \( f \) on \( G \) by \( Tf = \{ \sum_0^\infty |S_{\Lambda_k}f|^{p'} \}^{1/p'} \), then \( \|Tf\|_{p,p'} \leq C\|f\|_p \). Hence \( T \) can be extended to a bounded operator from \( L^p(G) \) into \( L^{p,p'}(G) \).

PROOF. For each \( k \geq 0, x \in G \) and \( f \in \mathcal{S}(G) \) we have

\[
|S_{\Lambda_k}f(x)| = |\psi_k \ast f(x)| \\
= \left| \int_G \gamma_k(x-y)\Delta_{n_k}(x-y)f(y)\,dy \right| \\
\leq \Delta_{n_k} \ast |f|(x) \\
\leq Mf(x).
\]

Thus,

\[
\sup_k |S_{\Lambda_k}f(x)| \leq Mf(x)
\]

so that the mapping

\[
f \rightarrow \sup_k |S_{\Lambda_k}f| \quad \text{is of weak type (1,1)}.
\]

We now choose \( \theta \) such that \( 1/p = 1-\theta/2 \), that is, \( \theta = 2(1-1/p) \); then \( 0 < \theta < 1 \). Let \( \Omega := \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \} \) and let \( f \in \mathcal{S}(G) \) such that \( \|f\|_p = 1 \). For \( z \in \Omega \) define the function \( f_z \) on \( G \) by

\[
f_z(x) = \begin{cases} \frac{f(x)}{|f(x)|}\delta^{(1-z/2)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0. \end{cases}
\]

Then \( f_z \in \mathcal{S}(G) \) for each \( z \in \Omega \). Moreover we have \( f_0 = f \), \( \|f_1\|_1 = 1 \) and \( \|f_{1+i}\|_2 = 1 \). For \( N \in \mathbb{N}, z \in \Omega \) and \( x \in G \) define the sequence \( \{ (T_N f_z)_k(x) \}_{k=0}^{\infty} \) by

\[
(T_N f_z)_k(x) = \begin{cases} S_{\Lambda_k}f_z(x) & \text{if } 0 \leq k \leq N \\ 0 & \text{if } k > N. \end{cases}
\]

Let \( [\ell^\infty, \ell^2]_\theta \) be the complex interpolation space. Then \( [\ell^\infty, \ell^2]_\theta = \ell^{p'} \) (see [Tr, 1.18.1, (12)]). For each \( x \in G \) define \( U_N f_\theta(x) \) by

\[
U_N f_\theta(x) := \left( \left( (T_N f_\theta)_k(x) \right)_{k=0}^{\infty} \right)_{[\ell^\infty, \ell^2]_\theta} \\
= \sum_{k=0}^N \left( |S_{\Lambda_k}f_\theta(x)|^{p'} \right)^{1/p'}.
\]
It follows from [Tr, 1.10.3, (9)] that

$$\log U_N f_\theta(x) \leq \int_{-\infty}^{\infty} P_0(\theta, t) \log \| (T_N f_{it})_k(x) \|_{L^\infty} dt$$

$$+ \int_{-\infty}^{\infty} P_1(\theta, t) \log \| (T_N f_{1+it})_k(x) \|_{L^2} dt,$$

(3)

where $P_0(\theta, t) \geq 0$, $P_1(\theta, t) \geq 0$, $\int_{-\infty}^{\infty} P_0(\theta, t) dt = 1 - \theta$ and $\int_{-\infty}^{\infty} P_1(\theta, t) dt = \theta$.

Thus, taking exponentials in (3) we have

$$U_N f_\theta(x) \leq \left\{ \exp \left( \frac{1}{1 - \theta} \int_{-\infty}^{\infty} P_0(\theta, t) \log \| (T_N f_{it})_k(x) \|_{L^{1/2}} dt \right) \right\}^{(1-\theta)} \times \left\{ \exp \left( \frac{1}{\theta} \int_{-\infty}^{\infty} P_1(\theta, t) \log \| (T_N f_{1+it})_k(x) \|_{L^{1/2}} dt \right) \right\}^{\theta}.$$

It follows from Jensen’s inequality that

$$U_N f_\theta(x) \leq \{ H_{N,0}(x) \}^{(1-\theta)} \{ H_{N,1}(x) \}^\theta,$$

where

$$H_{N,0}(x) = \left( \frac{1}{1 - \theta} \int_{-\infty}^{\infty} P_0(\theta, t) \| (T_N f_{it})_k(x) \|_{L^{1/2}} dt \right)^2$$

and

$$H_{N,1}(x) = \left( \frac{1}{\theta} \int_{-\infty}^{\infty} P_1(\theta, t) \| (T_N f_{1+it})_k(x) \|_{L^{1/2}} dt \right)^2.$$

For each measurable subset $E$ of $G$ we have

$$\left( \frac{1}{\mu(E)} \int_E (U_N f_\theta(x))^{1/2} dx \right)^2 \leq \left( \frac{1}{\mu(E)} \int_E \{ H_{N,0}(x) \}^{(1-\theta)/2} \{ H_{N,1}(x) \}^{\theta/2} dx \right)^2 \leq \left( \frac{1}{\mu(E)} \int_E \{ H_{N,0}(x) \}^{1/2} dx \right)^{2(1-\theta)} \left( \frac{1}{\mu(E)} \int_E \{ H_{N,1}(x) \}^{1/2} dx \right)^{2\theta}.$$

It follows that for $y > 0$

$$(U_N f_\theta)**(y) = \sup_{y \leq \mu(E)} \left( \frac{1}{\mu(E)} \int_E (U_N f_\theta(x))^{1/2} dx \right)^2 \leq \{ H_{N,0}**(y) \}^{(1-\theta)} \{ H_{N,1}**(y) \}^\theta.$$
Since \((U_N f_0)^{**} = ((U_N f_0)^{*})^{*}\) we have, for \(1 < p < \infty\),
\[
\|U_N f_\theta\|_{p,p'} = \left( \frac{p'}{p} \int_0^\infty \left( t^{1/p} (U_N f_\theta)^{*}(t) \right)^{p'} dt \right)^{1/p'} \\
\leq \left( \frac{p'}{p} \int_0^\infty \left( t^{1/p} \{H^{**}_{N,0}(t)\}^{(1-\theta)} \{H^{**}_{N,1}(t)\}^{\theta} \right)^{p'} dt \right)^{1/p'} \\
= \left( \frac{p'}{p} \int_0^\infty \{t H^{**}_{N,0}(t)\}^{(1-\theta)p'} \{t^{1/2} H^{**}_{N,1}(t)\}^{\theta p'} dt \right)^{1/p'} \\
= \left( \frac{p'}{p} \int_0^\infty \{t H^{**}_{N,0}(t)\}^{(1-\theta)p'} \{t^{1/2} H^{**}_{N,1}(t)\}^{2} dt \right)^{1/p'},
\]
since \(\theta p' = 2(1 - 1/p)p' = 2\). By Hölder’s inequality we have
\[
\|U_N f_\theta\|_{p,p'} \leq \sup \{t H^{**}_{N,0}(t)\}^{(1-\theta)} \left( \frac{p'}{p} \int_0^\infty \{t^{1/2} H^{**}_{N,1}(t)\}^{2} dt \right)^{\theta/2} \\
\leq B \|H_{N,0}\|_{1,\infty}^{(1-\theta)} \|H_{N,1}\|_2^{\theta},
\]
where we use \(1/p' = \theta/2\) in the first inequality.

We shall estimate \(\|H_{N,0}\|_{1,\infty}\). For \(y > 0\) we have
\[
H^{**}_{N,0}(y) = \sup_{y \leq \mu(E)} \left( \frac{1}{\mu(E)} \int_E |H_{N,0}(x)|^{1/2} d\mu(x) \right)^2 \\
= \sup_{y \leq \mu(E)} \left[ \frac{1}{\mu(E)} \int_E \left( \frac{1}{1 - \theta} \int_{-\infty}^\infty P_0(\theta, t) \|\{(T_N f_{it})_k(x)\}\|_2^{1/2} dt \right) d\mu(x) \right]^2 \\
= \left[ \frac{1}{(1 - \theta)} \int_{-\infty}^\infty P_0(\theta, t) \left( \sup_{y \leq \mu(E)} \frac{1}{\mu(E)} \int_E \|\{(T_N f_{it})_k(x)\}\|_2^{1/2} d\mu(x) \right) dt \right]^2,
\]
where the last equality follows from Fubini’s theorem. Now
\[
\|\{(T_N f_{it})_k(x)\}\|_2 = \sup_{0 \leq k \leq N} |S_{\Lambda_k} f_{it}(x)| \leq \sup_{0 \leq k} |S_{\Lambda_k} f_{it}(x)| := F_{it}(x).
\]
Therefore
\[
H^{**}_{N,0}(y) \leq \left[ \frac{1}{(1 - \theta)} \int_{-\infty}^\infty P_0(\theta, t) \left( \sup_{y \leq \mu(E)} \frac{1}{\mu(E)} \int_E (F_{it}(x))^{1/2} d\mu(x) \right) dt \right]^2 \\
\leq \left[ \frac{1}{(1 - \theta)} \int_{-\infty}^\infty P_0(\theta, t) ((F_{it})^{**}(y))^{1/2} dt \right]^2 \\
\leq (F_{it})^{**}(y).
\]
Consequently we have
\[ \|H_{N,0}\|_{1,\infty} \leq \|F_{it}\|_{1,\infty} = \|\{S_{\Lambda_k} f_{it}\}\|_{1,\infty} \leq C \|f_{it}\|_1 = C, \]
where the last inequality follows from (2). Similarly we have
\[ \|H_{N,1}\|_2 \leq C \left( \frac{1}{\theta} \int_{-\infty}^{\infty} P_1(\theta, t) \|f_{1+it}\|_2^{1/2} dt \right)^2 \]
\[ \leq C \|\Delta f_{1+it}\|_2 \]
\[ = C \|f_{1+it}\|_2 \quad \text{(using (1))} \]
\[ = C. \]

It follows that \( \|U_N f_\theta\|_{p,p'} \leq C \) if \( \|f\|_p = 1 \). Since \( \|T f\|_{p,p'} = \lim_{N \to \infty} \|U_N f_\theta\|_{p,p'} \), we have \( \|T f\|_{p,p'} \leq C \|f\|_p \) for \( f \in \mathcal{S}'(G) \). Now \( \mathcal{S}'(G) \) is dense in \( L^p(G) \) and so \( T \) can be extended to all functions in \( L^p(G) \) and our proof is complete. \( \Box \)

We observe that inequality (**) in the proof of Theorem 2 above shows that for each \( r > 1 \) we have
\[ \left\| \sup_k |S_{\Lambda_k} f| \right\|_r \leq \|M f\|_r \leq C \|f\|_r. \]
Interpolation between (1) and (4) yields the following theorem.

**Theorem 3.** Let \( 1 < p < 2 \) and let \( \{\Lambda_k\}_{k=0}^\infty \) be as in Theorem 2. If \( s > p' \), then
\[ \left\| \sum_{k=0}^{\infty} \left( |S_{\Lambda_k} f|^s \right)^{1/s} \right\|_p \leq C \|f\|_p. \]

Another result we can derive from inequality (*) in the proof of Theorem 2 is the following theorem.

**Theorem 4.** Assume \( \{\Lambda_k\}_{k=0}^\infty = \{\gamma_k + \Gamma_{n_0}\}_{k=0}^\infty \) for some fixed \( n_0 \) (i.e. we have a partition of \( \Gamma \) into the cosets of a fixed subgroup \( \Gamma_{n_0} \) of \( \Gamma \)). Then
\[ \left\| \sum_{k=0}^{\infty} \left( |S_{\Lambda_k} f|^{p'} \right)^{1/p'} \right\|_p \leq C \|f\|_p \]

**Proof.** According to (*) in the proof of Theorem 2, we have for every \( k \geq 0 \),
\[ |S_{\Lambda_k} f(x)| \leq \Delta_{n_0} \ast |f|(x), \]
so that
\[ \left\| \sup_k |S_{\Lambda_k} f| \right\|_1 \leq \|\Delta_{n_0} \ast |f|\|_1 \leq \|\Delta_{n_0}\|_1 \|f\|_1 = \|f\|_1. \]
Interpolation between (1) and (6) yields (5). \( \Box \)
C. W. Onneweer and T. S. Quek

REMARK. Note that a slight generalization of this result can be obtained by considering partitions \( \{ \Lambda_k \}_{k=0}^{\infty} = \{ \gamma_k + \Gamma_n \}_{k=0}^{\infty} \) satisfying the condition \( \sup_k \lambda(\gamma_k + \Gamma_n) = \sup_k m_{n_k} = m_{n_k} \) for some \( n_\alpha \in \mathbb{Z} \). In this case we have for each \( k \geq 0 \),

\[
|S_{\Lambda_k} f(x)| \leq \Delta_{n_k} \ast |f|(x)
\]

so that

\[
\sup_k |S_{\Lambda_k} f(x)| \leq \sum_{\ell=0}^{\alpha} \Delta_{n_\ell} \ast |f|(x).
\]

Therefore,

\[
\left\| \sup_k |S_{\Lambda_k} f| \right\|_1 \leq \sum_{\ell=0}^{\alpha} \| \Delta_{n_\ell} \ast |f| \|_1 \leq C \| f \|_1,
\]

yielding again (5).

5. Multipliers on \( L^p(G) \)

In [OQ] we considered the decomposition of \( \Gamma \) into disjoint sets \( \Gamma_{k+1} \setminus \Gamma_k \) and in [OQ, Theorem 2.1] the following multiplier theorem was proved.

THEOREM OQ. Let \( 1 < p < \infty \) and let \( \{ \phi_k \}_{k=-\infty}^{\infty} \in \ell^s(\mathcal{M}(L^p(G))) \) for some \( 0 < s \leq |2p/(2 - p)| \). If \( \phi := \sum_{k=-\infty}^{\infty} \phi_k \chi_{\Gamma_{k+1} \setminus \Gamma_k} \in L^\infty(\Gamma) \) then \( \phi \in \mathcal{M}(L^p(G)) \).

As an application of Theorem 1 we prove a comparable result for decompositions of \( \Gamma \) as considered in the present paper, see Theorem 5. Our proof was motivated by [CFF, Theorem 2] and is similar to that of [OQ, Theorem 2.1]. We shall discuss the sharpness of Theorem 5 in Theorem 8.

THEOREM 5. Let \( \{ \Lambda_k \}_{k=0}^{\infty} = \{ \gamma_k + \Gamma_n \}_{k=0}^{\infty} \) be as in Theorem 1 and let \( 1 < p < \infty \). Let \( \{ \phi_k \}_{k=0}^{\infty} \in \ell^s(\mathcal{M}(L^p(G))) \) for \( s = |p/(2 - p)| \) and assume \( \phi := \sum_{k=0}^{\infty} \phi_\Lambda \chi_{\Lambda_k} \in L^\infty(\Gamma) \). Then \( \phi \in \mathcal{M}(L^p(G)) \).

PROOF. We may assume that \( 2 < p < \infty \) and that \( s = p/(p - 2) \). Take any \( f \in \mathcal{S}(G) \). A direct computation for the cases \( p = 2 \) and \( p = \infty \), followed by an interpolation argument shows that the following inequality holds:

\[
\| (\hat{\phi} \hat{f})^\vee \|_{p'} = \left\| \sum_k \psi_k \ast (\phi_k \chi_{\Lambda_k} \hat{f})^\vee \right\|_{p'} \leq C \left\| \sum_k (\phi_k \chi_{\Lambda_k} \hat{f})^\vee \right\|_{p'}.
\]
Therefore,
\[
\| (\phi \hat{f})^\nu \|_{p}^{p'} \leq C \sum_{k} \| \phi_{k, \mathcal{M}(L^{p})} \|_{p}^{p'} \| S_{\Lambda_{k}} f \|_{p}^{p'}
\]
\[
\leq C \left( \sum_{k} \| \phi_{k, \mathcal{M}(L^{p})} \|_{p'}^{p'} \right)^{2-p'} \left( \sum_{k} \| S_{\Lambda_{k}} f \|_{p}^{p} \right)^{p'/p}
\]
\[
= C \left( \sum_{k} \| \phi_{k, \mathcal{M}(L^{p})} \|_{p'}^{p'/s} \right)^{p'/s} \left( \int_{G} \sum_{k} |S_{\Lambda_{k}} f(x)|^{p} \, dx \right)^{p'/p}
\]
\[
\leq C \left( \sum_{k} \| \phi_{k, \mathcal{M}(L^{p})} \|_{p'}^{p'/s} \right)^{p'/s} \left( \int_{G} \left\{ \sum_{k} |S_{\Lambda_{k}} f(x)|^{2} \right\}^{p/2} \, dx \right)^{p'/p}
\]
\[
\leq C \| f \|_{p}^{p'}
\]

where the penultimate inequality holds because \( 2 < p \), while the final inequality follows from Theorem 1. \( \square \)

As an additional application of Theorems 1 and 2 we have

**THEOREM 6.** Let \( \{ \Lambda_{k} \}_{k=0}^{\infty} \) be a decomposition of \( \Gamma \) as in Theorem 1.

(i) If \( \{ a_{k} \}_{k=0}^{\infty} \in \ell^{2} \), then \( \sum_{k=0}^{\infty} a_{k} \chi_{\Lambda_{k}} \) is a multiplier on \( L^{p}(G) \) for \( 1 < p < \infty \).

(ii) If \( \{ a_{k} \}_{k=0}^{\infty} \in \ell^{s} \) for some \( s > 2 \), then \( \sum_{k=0}^{\infty} a_{k} \chi_{\Lambda_{k}} \) is a multiplier from \( L^{p}(G) \) into \( L^{p,p'}(G) \) for \( 2s/(2+s) \leq p \leq 2 \).

**PROOF.** (i) It follows from Theorem 1 that for \( 2 \leq p < \infty \) we have
\[
\left\| \sum_{k=0}^{\infty} (a_{k} \chi_{\Lambda_{k}} \hat{f})^{\nu} \right\|_{p} \leq \left\| \sum_{k=0}^{\infty} |a_{k}|^{2} \right\|^{1/2} \left\| \sum_{k=0}^{\infty} |S_{\Lambda_{k}} f|^{2} \right\|^{1/2} \leq C \| f \|_{p}
\]

Hence \( \sum_{k=0}^{\infty} a_{k} \chi_{\Lambda_{k}} \) is a multiplier on \( L^{p}(G) \) for \( 2 \leq p < \infty \). The case \( 1 < p < 2 \) follows from duality.

(ii) Applying real interpolation (see [Tr, 1.18.6, Theorem 2]) to the inequalities obtained from the cases \( p = 2 \) and \( p = r^{*} \) for some \( r^{*} > r \) of Theorem 1, we obtain
\[
\left\| \left\{ \sum_{k=0}^{\infty} |S_{\Lambda_{k}} f|^{2} \right\}^{1/2} \right\|_{r,q} \leq C \| f \|_{r,q}
\]

for \( 2 < r < \infty \) and \( 1 \leq q < \infty \). Also, an argument as in [St, Chapter IV, 5.3.1] shows that for all \( f, g \in \mathcal{S}(G) \)
\[
\int_{G} f(x)g(x) \, dx = \sum_{k=0}^{\infty} \int_{G} S_{\Lambda_{k}} f(x) \overline{S_{\Lambda_{k}} g(x)} \, dx.
\]
Next, a standard argument using (7), (8) and the converse of Hölder’s inequality for Lorentz spaces shows that for \(1 < p < 2\)

\[
\|f\|_{p,p'} \leq C \left\| \left( \sum_{k=0}^{\infty} |S_{\Lambda_k} f|^2 \right)^{1/2} \right\|_{p,p'}.
\]

Finally, set \(t = 2s/(2 + s)\); using inequality (9), Hölder’s inequality and Theorem 2 (see the proof in [CFF, p.341]) shows that

\[
\left\| \sum_{k=0}^{\infty} (a_k \chi_{\Lambda_k} \hat{f})^\wedge \right\|_{t,t'} \leq C \left\| \left( \sum_{k=0}^{\infty} |a_k S_{\Lambda_k} f|^2 \right)^{1/2} \right\|_{t,t'} \leq C \|f\|_t.
\]

Hence \(\sum_{k=0}^{\infty} a_k \chi_{\Lambda_k}\) is a multiplier from \(L^t(G)\) into \(L^{t,t'}(G)\). The result now follows because \(\sum_{k=0}^{\infty} a_k \chi_{\Lambda_k}\) is also a multiplier on \(L^2(G)\).

\[
\text{6. Sharpness of certain results}
\]

The following theorem shows that Theorem 2 is sharp in a certain sense.

**THEOREM 7.** Let \(1 < p < 2\) and let \(s < p'\). There exists a decomposition \(\{\Lambda_k\}_{k=0}^{\infty}\) of \(\Gamma\) into mutually disjoint cosets of various subgroups of \(\Gamma\) such that the mapping \(f \to \left\{ \sum_{k=0}^{\infty} |S_{\Lambda_k} f|^2 \right\}^{1/s}\) is not bounded from \(L^p(G)\) to \(L^{p,p'}(G)\).

**PROOF.** Take \(\{\Lambda_k\}_{k=0}^{\infty} = \{\gamma_k + \Gamma_0\}_{k=0}^{\infty}\), that is, partition \(\Gamma\) into the cosets of \(\Gamma_0\) and choose the \(\gamma_k\) in such a way that for each \(l \geq 0\), we have

\[
\bigcup_{0 \leq k < m_l} \gamma_k + \Gamma_0 = \Gamma_l.
\]

Next, for \(l \geq 0\), let \(f_l(x) = \Delta_l(x)\), so that \(\|f_l\|_p = (m_l)^{1/p'}\) and \((f_l)^\wedge(\gamma) = \chi_{\Gamma_l}(\gamma)\). Then

\[
S_{\Lambda_k} f_l(x) = \begin{cases} 
\chi_{\Gamma_0}(x) \gamma_k(x) & \text{if } k < m_l \\
0 & \text{if } k \geq m_l.
\end{cases}
\]

Therefore,

\[
\left\| \left( \sum_{k=0}^{m_l-1} |S_{\Lambda_k} f_l|^s \right)^{1/s} \right\|_{p,p'}^{*} = (m_l)^{1/s}.
\]
If there were a constant $C$ such that
\[
\left\| \left( \sum_{k=0}^{m_l-1} |S_{\Lambda_k} f_l(x)|^r \right)^{1/r} \right\|_{L^p}^{1/s} \leq C \| f_l \|_p,
\]
then we would have $(m_l)^{1/s} \leq C (m_l)^{1-1/p}$ for all $l \geq 0$. But this is impossible because $s < p'$.

Theorem 7 has the following obvious corollary which shows that Theorem 1 is not necessarily true if $1 < p < 2$.

**Corollary.** Let $1 < p < 2$. Then there exists a decomposition $\{\Lambda_k\}_{k=0}^\infty$ of $\Gamma$ into mutually disjoint cosets of various subgroups of $\Gamma$ such that the mapping $f \to \Delta f$ is not bounded on $L^p(G)$, where $\Delta f$ is as defined in Theorem 1.

Next we prove the sharpness of Theorem 5. The example constructed in the proof of Theorem 8 below is analogous to [CFF, Example 2].

**Theorem 8.** Let $1 < p < \infty$ and assume that $q > s = |p/(2 - p)|$. Then there exists a decomposition $\{\Lambda_k\}_{k=0}^\infty$ of $\Gamma$ as in Theorem 1 and functions $\{\phi_k\} \in \mathcal{M}(L^p(G))$ such that

(a) $\text{supp } \phi_k = \Lambda_k$ for all non-negative integers $k$,

(b) $\{\phi_k\} \in l^q(\mathcal{M}(L^p(G)))$,

(c) if $\phi := \sum_{k=0}^\infty \phi_k$ then $\phi \in L^\infty(\Gamma)$ and $\phi \notin \mathcal{M}(L^p(G))$.

**Proof.** We assume that $1 < p < 2$ so that $s = p/(2 - p)$. Take $\{\Lambda_k\}_{k=0}^\infty = \{\gamma_k + \Gamma_0\}_{k=0}^\infty$ and choose the $\gamma_k$ so that for each $l \geq 0$, we have
\[
\bigcup_{0 \leq k < m_l} \gamma_k + \Gamma_0 = \Gamma_l.
\]
Choose $\alpha$ so that $1/q < \alpha < 1/s$. For each $k \geq 0$ choose an $x_k \in G_{-k} \setminus G_{-k+1}$ and define the functions $\phi_k : \Gamma \to \mathbb{C}$ by
\[
\phi_k(\gamma) = (k + 1)^{-\alpha} \gamma(x_k) \chi_{\Lambda_k}(\gamma).
\]
Then we have $(\phi_k)^\gamma(x) = (k + 1)^{-\alpha} \gamma(x) \chi_{G_0}(x - x_k)$, so that $\|\phi_k\|_{\mathcal{M}(L^p)} \leq \|\phi_k\|_1 = (k + 1)^{-\alpha}$. Hence the sequence $\{\phi_k\}$ satisfies conditions (a) and (b).

Moreover, if we define $\phi := \sum_{k=0}^\infty \phi_k$, then it can be shown as in [OQ, Theorem 2.2] that $\phi \notin \mathcal{M}(L^p(G))$. This completes the proof of Theorem 8.

Our last result shows that Theorem 6 is also best possible in a certain sense.
THEOREM 9. Let $G$ be the dyadic group. Let $2 < s < \infty$ and let $p = 2s/(2 + s)$. Then there exists a sequence $\{a_k\}_{k=1}^{\infty} \in \ell^s$ and a decomposition of $\Gamma$ as in Theorem 1 so that $\sum_{k \in \mathbb{N}} a_k \chi_{\Lambda_k}$ is not a multiplier from $L^s(G)$ into $L^{r,r}(G)$ for any $r$ such that $1 < r < p$.

PROOF. Following [GI, Example 5.2], we construct Rudin-Shapiro-like polynomials on $G$ as follows:

For $0 \leq n$, fix $\gamma_0^n$ in $\Gamma_{2n+2} \setminus \Gamma_{2n+1}$ and let

$$\rho_0^n = \sigma_0^n = \chi_{\gamma_0^n}.$$ 

Next, for $k = 1, \ldots, n + 1$, set

$$\rho_k^n = \rho_{k-1}^n + \gamma_k^n \sigma_{k-1}^n,$$

and

$$\sigma_k^n = \rho_{k-1}^n - \gamma_k^n \sigma_{k-1}^n,$$

where $\gamma_k^n$ are chosen from $\Gamma_{2n+1}$ such that $(\rho_k^n)^\wedge$ and $(\sigma_k^n)^\wedge$ are both constant and non-zero on precisely $2^k$ cosets of $\Gamma_n$ in $\Gamma_{2n+2} \setminus \Gamma_{2n+1}$. Now define $\Theta$ on $\Gamma$ by

$$\Theta(\gamma) = \begin{cases} 
\text{sgn}(\rho_{n+1}^n)^\wedge(\gamma) & \text{if } \gamma \in \Gamma_{2n+2} \setminus \Gamma_{2n+1}, \ n \geq 0 \\
0 & \text{otherwise}.
\end{cases}$$

Choose $q$ such that $r < q < p$ and choose $\alpha$ so that $q < 2/(2 - \alpha) < p$; then $0 < \alpha < 1$. Define $\Phi$ on $\Gamma$ by

$$\Phi(\gamma) = \sum_{n \in \mathbb{N}} 2^{(\alpha - 1)n/2} \chi_{\Gamma_{2n} \setminus \Gamma_{2n-1}}(\gamma) \Theta(\gamma).$$

Note that for $n \geq 1$, $\Phi(\gamma)$ is constant ($= \pm 2^{(\alpha - 1)n/2}$) on the $2^n$ cosets of $\Gamma_{n-1}$ in $\Gamma_{2n} \setminus \Gamma_{2n-1}$ and is zero elsewhere. Denote the $2^n$ cosets of $\Gamma_{n-1}$ in $\Gamma_{2n} \setminus \Gamma_{2n-1}$ by $\Lambda_{(2n,k)}$ for $k = 1, \ldots, 2^n$. Now define the sequence $\{a_{(2n,k)}\}$ such that

$$|a_{(2n,k)}| = 2^{(\alpha - 1)n/2} \quad \text{for } n \in \mathbb{N}, \ k = 1, \ldots, 2^n$$

and satisfying

$$\sum_{n \in \mathbb{N}} \sum_{k=1}^{2^n} a_{(2n,k)} \chi_{\Lambda_{(2n,k)}}(\gamma) = \sum_{n \in \mathbb{N}} 2^{(\alpha - 1)n/2} \chi_{\Gamma_{2n} \setminus \Gamma_{2n-1}}(\gamma) \Theta(\gamma).$$
It is easy to see that
\[ \sum_{n \in \mathbb{N}} \sum_{k=1}^{2^n} |a_{(2n,k)}|^s < \infty. \]

Now suppose \( \Phi \) were a multiplier from \( L^r(G) \) to \( L^{r',}(G) \), then \( \Phi \) would be a multiplier on \( L^q(G) \) because \( \Phi \) is a multiplier on \( L^2(G) \) and \( r < q < p < 2 \). But by [GI, Example 5.2] \( \Phi \) is not a multiplier on \( L^q(G) \). Hence we have a contradiction. \( \Box \)

References


Department of Mathematics
University of New Mexico
Albuquerque, NM 87131
USA
e-mail: onneweer@math.unm.edu

Department of Mathematics
National University of Singapore
Singapore 119260
Republic of Singapore
e-mail: matqts@leonis.nus.sg