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STRONG SUMMABILITY FOR THE MARCINKIEWICZ MEANS IN THE INTEGRAL METRIC AND RELATED QUESTIONS

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Abstract

The inequality of strong summability for the Marcinkiewicz means of multiply Fourier series is proved. The inequalities of strong summability with gaps for the different classes of integrable functions are established. The Bernstein inequality for the fractional derivative of analytic polynomials is proved.

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1. Marcinkiewicz means in $L^1(Q^d)$

Let $f(x)$ be a periodic function of d variables defined on the unit cube $Q^d = [-\pi, \pi]^d$. Every function $f \in L^1(Q^d)$ can be expanded in the Fourier series

$$f \simeq \sum_k \hat{f}(k) e^{i(k,x)}$$

where $k = (k_1, \dots, k_d)$ is a multi-index, and $x = (x_1, \dots, x_d)$ is a vector. Let $\|k\|_\infty = \sup_{j=1,\dots,d} |k_j|$. Let

$$S_m(f;x) = \sum_{\|k\|_\infty \le m} \hat{f}(k) e^{i(k,x)}$$

denote the cubic partial sums.

For every $f \in L^\infty(Q^d)$ the following inequality of strong summability for the Marcinkiewicz means is well known [12]: (the classical one-dimensional result can be found, for example, in [17, Chapter 13]):

$$\frac{1}{N}\sum_{m=0}^{N}|S_m(f;x)| \le C\|f\|_\infty.$$

In this paper we consider strong summability for the Marcinkiewicz means in the integral metric. The first one-dimensional result was proved by Smith [15] for $f \in H^1(Q)$. Its generalization for $f \in H^p(Q)$, $0 < p < 1$, for the Riesz means of 'critical' order together with an estimate of the approximation rate, were given in [7]. A new proof was proposed in [4] where the exact order of approximation was found. The two-dimensional result for the rectangular partial sums with bounded ratio of sides was obtained in [16], but it is still unclear how to generalize this result to the multidimensional case. We use here the idea of 'harmonic' proof of [4] to obtain the multidimensional result for the cubic partial sums.

We need some additional notation. If $\alpha = (\alpha_1, \dots, \alpha_d)$ is a vector then we write $m_\alpha(u) = \operatorname{sign}(u, \alpha)$, and denote by H_α the operator of the Hilbert transform

$$H_\alpha : f(x) \to \sum_k m_\alpha(k)\hat{f}(k)e^{i(k,x)}.$$

Let $\{e_j\}_1^d$ be the basis vectors. We introduce two operators

$$H_j = \prod_{i\ne j}(H_{e_j+e_i} + H_{e_j-e_i}); \qquad H_j' = H_{e_j}H_j.$$

THEOREM 1.1. *There exists an absolute constant $C > 0$ such that*

$$\frac{1}{\log N}\sum_{m=1}^{N}\frac{1}{m}\|S_m(f;x)\|_1 \le C\sum_{j=1}^{d}\left(\|H_j f\|_1 + \|H_j' f\|_1\right).$$

PROOF. It is based on the Hardy inequality (see for example [17])

$$\sum_{m=0}^{N}\frac{|c_m|}{m+1} \le C\int_{-\pi}^{\pi}\left|\sum_{m=0}^{N}c_m e^{imt}\right| dt.$$

Before applying the Hardy inequality to the sum $\sum_{m=1}^{N}\frac{1}{m}|S_m(f;x)|$ we will transform it slightly. The de La Vallee-Poussin means $V_R(f;x) = f(x) * V_R(x)$ are the convolution of $f(x)$ with the kernel $V_R(x) = \prod_{j=1}^{d} V_R(x_j)$ where

$$V_R(x_j) = \sum_{k=0}^{\infty} v(k/R)\cos kx_j,$$

and $v(t)$ is any infinitely differentiable even function such that

$$v(t)=\begin{cases}1, & \text{if} \quad |t|<1\\ 0, & \text{if} \quad |t|>2.\end{cases}$$

Because $m \le N$ we can replace $S_m(f;x)$ by $S_m(V_N(f);x)$. Then

$$\begin{aligned}\sum_{m=1}^{N}\frac{1}{m}|S_m(f;x)| &= \sum_{m=1}^{N}\frac{1}{m}|S_m(V_N(f;x))|\\ &\le \sum_{m=1}^{2N}\frac{1}{m}|S_m(V_N(f;x))| \le C\int_{-\pi}^{\pi}\left|\sum_{m=0}^{2N}e^{imt}S_m(V_N(f;x))\right|\,dt.\end{aligned}$$

Integrating both sides with respect to x we obtain

$$\begin{aligned}&\sum_{m=1}^{N}\frac{1}{m}\|S_m(f;x)\|_1\\ &\quad\le C\int_{Q^d}dx\int_{-\pi}^{\pi}\left|\sum_{m=0}^{2N}e^{imt}\sum_{s=1}^{m}\sum_{\|k\|_\infty=s}\prod_{j=1}^{d}v\left(\frac{k_j}{N}\right)\hat{f}(k)e^{i(k,x)}\right|\,dt\\ &\quad=\int_{-\pi}^{\pi}dt\int_{Q^d}\left|\sum_{s=1}^{2N}\sum_{\|k\|_\infty=s}\prod_{j=1}^{d}v\left(\frac{k_j}{N}\right)\hat{f}(k)e^{i(k,x)}\sum_{m=s}^{2N}e^{imt}\right|\,dx\\ &\quad\le\int_{-\pi}^{\pi}\left|\sum_{m=1}^{2N}e^{imt}\right|\,dt\int_{Q^d}\left|\sum_{s=1}^{2N}\sum_{\|k\|_\infty=s}\prod_{j=1}^{d}v\left(\frac{k_j}{N}\right)\hat{f}(k)e^{i(k,x)}\right|\,dx\\ &\qquad+\int_{-\pi}^{\pi}\frac{dt}{|e^{it}-1|}\int_{Q^d}\left|\sum_{s=1}^{2N}\sum_{\|k\|_\infty=s}\prod_{j=1}^{d}v\left(\frac{k_j}{N}\right)\hat{f}(k)e^{i(k,x)}(e^{ist}-1)\right|\,dx.\end{aligned}$$

The first integral can be easily estimated by $O(\log N\|V_N(f;x)\|_1)$. The well known result for the de La Vallee-Poussin means (see for example [17])

$$\|V_N(f;x)\|_1 \le C\|f\|_1$$

completes the estimate of the first integral.

For the second integral, we transform the interior sum to the form

$$\frac{1}{2}\sum_{j=1}^{d}(H_j^{'}+H_j)V_N(f;x+te_j)+(H_j^{'}-H_j)V_N(f;x-te_j)-2H_j^{'}V_N(f;x),$$

if d is even, and to the form

$$\frac{1}{2}\sum_{j=1}^{d}(H_j+H_j^{'})V_N(f;x+te_j)+(H_j-H_j^{'})V_N(f;x-te_j)-2H_jV_N(f;x),$$

if d is odd. Therefore the second integral is estimated by

$$\sum_{j=1}^{d}\int_0^{\pi}\|V_N(H_j f;x+te_j)-V_N(H_j f;x-te_j)\|_1\frac{dt}{t}$$
$$+\sum_{j=1}^{d}\int_0^{\pi}\|V_N(H_j^{'} f;x+te_j)-2V_N(H_j^{'} f;x)+V_N(H_j^{'} f;x-te_j)\|_1\frac{dt}{t}$$

in the case of an even dimension, and by the symmetrical expression in the case of an odd dimension.

By the Bernstein inequality in the space L^1,

$$\|V_N(H_j f;x+te_j)-V_N(H_j f;x-te_j)\|_1\le\min(Nt,2)\|(H_j f;x)\|_1.$$

Hence

$$\int_0^{\pi}\|V_N(H_j f;x+te_j)-V_N(H_j f;x-te_j)\|_1\frac{dt}{t}\le C\log N\|V_N(H_j f;x)|_1.$$

The regularity of the de La Vallee-Poussin means completes the proof. □

2. Strong summability with gaps

In this section we consider several examples of strong summability of the lacunary sequence of partial sums using the same idea of harmonic proof and the Paley inequality or its variations.

Let $f\in H^1(Q)$, and $\{n_k\}$ be a lacunary sequence ($n_{k+1}/n_k>q>1$). Then the classical Paley inequality (see for example [17]) is

$$\left\{\sum_{k=1}^{\infty}\left|\hat{f}(n_k)\right|^2\right\}^{1/2}\le C\|f\|_{H^1}.$$

It forms the basis for our next result.

THEOREM 2.1. *There exists an absolute constant $C>0$ such that*

$$\int_Q\left(\sum_{k=1}^{N}|S_{2^k}(f;x)|^2\right)^{1/2}dx\le CN\|f\|_{H^1}.$$

PROOF. The proof of the result follows in exactly the same way as the proof of Theorem 1.1, and we omit it. □

REMARK 2.2. This inequality is exact in order. Indeed, leaving only the last item in the left-hand part we obtain the Lebesgue inequality:

$$\int_Q |S_{2^N}(f;x)|\, dx \le CN\|f\|_{H^1}.$$

REMARK 2.3. For the continuous functions $f \in C(Q)$ the analogous result was obtained in [1] (See also [3], where the general problem was considered).

COROLLARY 2.4. *There exists an absolute constant* $C > 0$ *such that*

$$\left\{\sum_{k=1}^{N} \|S_{2^k}(f;x)\|_1^2\right\}^{1/2} \le CN\|f\|_{H^1}.$$

PROOF.

$$\int_Q \left(\sum_{k=1}^{N} |S_{2^k}(f;x)|^2\right)^{1/2} dx = \int_Q \sup_{\|b\|_{l^2}\le 1} \sum_{k=1}^{N} |b_k S_{2^k}(f;x)|\, dx$$
$$\ge \sup_{\|b\|_{l^2}\le 1} \sum_{k=1}^{N} |b_k|\|S_{2^k}(f;x)\|_1 = \left\{\sum_{k=1}^{N} \|S_{2^k}(f;x)\|_1^2\right\}^{1/2}.$$

Theorem 2.1 completes the proof. □

REMARK 2.5. Let $d(N)$ be the cardinality of the set $\{k : 1 \le k \le N,\ \|S_{2^k}(f;x)\|_1 \ge Ck\|f\|_{H^1}\}$. Then

$$d(N) \le C \log N.$$

Indeed, Corollary 2.4 shows that each interval $[N, 2N]$ can contain only a finite number (independent of N) of partial sums with the prescribed property. This implies the estimate of cardinality.

This situation is completely different from the space L^∞ where according to [6] there exists a bounded function f such that, for every k, $\|S_k(f;x)\|_\infty \ge C \log k$.

THEOREM 2.6. *Let* $0 < p < 1$. *Then there exists a constant* $C_p > 0$ *which depends only on* p *such that*

$$\left(\sum_{k=1}^{\infty} \|S_{2^k}(f;x)\|_p^p 2^{k(p-1)}\right)^{1/p} \le C_p\|f\|_{H^p}.$$

PROOF. The proof of this result follows by the same method but it is based on the following inequality [8]

$$\left(\sum_{k=1}^{\infty} \left| \hat{f}(2^k) \right|^p 2^{k(p-1)} \right)^{1/p} \leq C \| f \|_{H^p}. \qquad \square$$

For $0 < p \leq 1$ let $H^{p,\infty}$ denote the class of functions $f \in H^{p/2}$ such that

$$\| f \|_{p,\infty} = \sup_{A>0} Am\{x : |f(x)| > A\}^{1/p} < \infty$$

(see for example [1]).

THEOREM 2.7. *There exists an absolute constant $C > 0$ such that*

$$\left(\sum_{k=1}^{N} \| S_{2^k}(f; x) \|_1^2 \right)^{1/2} \leq C N^2 \| f \|_{H^{1,\infty}}.$$

We need the following lemmas.

LEMMA 2.8. *Let $P_n(e^{ix}) = \sum_{k=0}^{n} c_k e^{ikx}$ be an analytic polynomial. Then for $r = 1 - 1/n$*

$$\| P_n(e^{ix}) \|_1 \leq C \| P_n(re^{ix}) \|_1.$$

This lemma is an easy corollary of the Bernstein inequality (see for example [17]).

LEMMA 2.9 ([1]). *Let $f \in H^{1,\infty}$, $0 < r < 1$. Then*

$$\int_Q |f(re^{ix})| dx \leq C \| f \|_{H^{1,\infty}} \log \frac{2}{1-r}.$$

PROOF OF THEOREM 2.7. Applying Lemma 2.8 to each item of the sum

$$\left(\sum_{k=1}^{N} \| S_{2^k}(f; x) \|_1^2 \right)^{1/2},$$

using Corollary 2.4, and Lemma 2.9 with $r = 1 - 1/2^N$, we obtain the result. $\square$

The well-known Zygmund inequality (see for example [17])

$$\left\{ \sum_{k=1}^{\infty} \left| \hat{f}(n_k) \right|^2 \right\}^{1/2} \leq C \| f \|_{L \operatorname{Log}^{1/2} L} \qquad \frac{n_{k+1}}{n_k} > q > 1$$

formally resembles the Paley inequality but it brings nothing new, and the best result can be obtained by the direct estimates of the partial sums. It shows that the Paley and Zygmund inequalities are of different natures. We formulate the corresponding result in general form.

PROPOSITION 2.10. *For every* α, $0 < \alpha \le 1$, *there exists an absolute constant* $C > 0$ *such that*

$$\left\{ \sum_{k=1}^{N} \| S_{2^k}(f;x) \|_1^2 \right\}^{1/2} \le C N^{\frac{3}{2}-\alpha} \| f \|_{L \operatorname{Log}^\alpha L}.$$

PROOF. Because the partial sum operator is bounded from $L \operatorname{Log} L$ to L^1 (see [17]) we have

$$\| S_k(f;x) \|_1 = \| S_k(V_{2k}(f;x)) \|_1 \le \| V_{2k}(f;x) \|_{L \operatorname{Log} L}$$

Using the different metric inequality

$$\| V_{2k} \|_{L \operatorname{Log} L} \le C \log^{1-\alpha} k \| V_{2k} \|_{L \operatorname{Log}^\alpha L}$$

[10], and the boundedness of the de La Vallee-Poussin operator in the space $L \operatorname{Log}^\alpha L$, we obtain the estimate

$$C \log^{1-\alpha} k \| V_{2k}(f;x) \|_{L \operatorname{Log}^\alpha L} \le C \log^{1-\alpha} k \| f \|_{L \operatorname{Log}^\alpha L}.$$

It remains to apply it to each term of the left-hand side. □

REMARK 2.11. The exactness of the inequality can be checked on the de La Vallee-Poussin kernel $V_{2^N}(x)$ of order 2^{N+1} if we take into account that $\| V_N(x) \|_{L \operatorname{Log}^\alpha L} \simeq \log^\alpha N$.

3. The Bernstein inequality

Let us consider a trigonometric polynomial $T_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ and its fractional derivative

$$T_n^{(\alpha)}(\theta) = \sum_{k=-n}^{n} (ik)^\alpha c_k e^{ik\theta} \qquad (0 < \alpha < \infty)$$

in the Weyl sense (see for example [17]). The classical Bernstein inequality

$$\| T_n^{(\alpha)} \|_{L^p} \le c(\alpha) n^\alpha \| T_n \|_{L^p}$$

is well known for $1 \le p \le \infty$. The detailed history can be found in [14]. Some anomalies were discovered for the metric L^p when $0 < p < 1$ [5]. We prove here that in classical Hardy spaces H^p $(0 < p < 1)$ the Bernstein inequality looks regular.

THEOREM 3.1. *Let* $\alpha > 0$, *and* $0 < p \le 1$. *Then, for every polynomial* $T_n(\theta) = \sum_{k=1}^n c_k e^{ik\theta}$,

$$\|T_n^{(\alpha)}(\theta)\|_{H^p} \le c(\alpha, p) n^\alpha \|T_n(\theta)\|_{H^p}.$$

PROOF. The polynomial $T_n^{(\alpha)}(\theta)$ can be represented by convolution $T_n(\theta) * K_n(\theta)$ with the kernel

$$K_n(\theta) = \sum_{k \in \mathbb{Z}} g_n(k) e^{ik\theta},$$

where the $g_n(t)$ is any infinitely differentiable function satisfying

$$g_n(k) = \begin{cases} 0 & k \le 0 \\ k^\alpha & 0 < k \le n \\ 0 & 2n \le k \end{cases}.$$

The function $g_n(t)$ is the multiplier from $H^p(\mathbb{R})$ to $H^p(\mathbb{R})$, with norm less than or equal to $c(\alpha, p) n^\alpha$ by the following result.

THEOREM 3.2 ([11, Section III.7, Theorem 7.30]). *Let s be a positive integer. Suppose m is a function on* $\mathbb{R}^d$ *satisfying*

$$\left(\frac{1}{R^d} \int_{R<|x|<2R} \left| D^\beta m(x) \right|^2 dx \right)^{1/2} \le A R^{-|\beta|}$$

for every multi-index β *such that* $0 \le |\beta| \le s$ *and every* $R > 0$, *with A independent of R and* β. *Then, for every p such that* $(s/d + 1/2)^{-1} < p \le 1$, *m is a multiplier on* $H^p(\mathbb{R}^d)$, *and there is a constant C independent of m and f such that*

$$\left\| (m \hat{f}) \right\|_{H^p(\mathbb{R}^d)} \le C A \| f \|_{H^p(\mathbb{R}^d)} .$$

Moving to the estimation of the multiplier norm $H^p(Q) \to H^p(Q)$ we follow [13, p. 159]. To prove that $g_n(k)$ is the multiplier $H^p(Q) \to H^p(Q)$ it is sufficient to show this for every p-atom a on the unit circle. (Consult [11] for characterization of H^p spaces in terms of atoms.) Now using the Poisson summation formula, we can write

$$\begin{aligned} 4\pi^2 \|a * K_n(\theta)\|_{L^p}^p &\le \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} a(u) \hat{g}_n(t - u + 2\pi l)\, du \right|^p dt \\ &= \int_{\mathbb{R}} \left| \int_{-\pi}^{\pi} a(u) \hat{g}_n(t-u)\, du \right|^p dt. \end{aligned}$$

If we extend a by 0 to the complement of $[-\pi, \pi]$ we obtain a p-atom for the real line and the last expression is nothing but $\|a * \hat{g}_n\|_{L^p(\mathbb{R})}^p$. This is bounded by $c(p, \alpha) n^\alpha \|a\|_{L^p}$ by the quoted theorem. □

REMARK 3.3. The question about the exact constant in the Bernstein inequality for the fractional derivative is much more difficult. For integer α the equality $c(\alpha, p) = 1$ was proved in the nice work by Arestov [2].

REMARK 3.4. The analogous Bernstein inequality holds also in the space $H^p(\mathbb{R})$ for entire functions of exponential type.

Fix any testing function $\phi(t)$ such that $\int_{\mathbb{R}} \phi(t)\,dt = 1$. The function f is said to be a tempered distribution of the class $\in H^p(\mathbb{R})$ if

$$u^+ = \sup_{t>0} |\phi_t * f| \in L^p$$

(see [9]). This definition is independent of ϕ and is equivalent to several others (see [9]). Let $\alpha > 0$ be a real number. We define the derivative $f^{(\alpha)}$ of order α as a convolution $f * \hat{g}_n$, where

$$g_n(t) = \begin{cases} 0 & t \le 0 \\ t^\alpha & 0 < t \le n \\ 0 & n \le t \end{cases} .$$

In the space $H^p(\mathbb{R})$ we consider entire functions of exponential type less than or equal to n, that is their spectrum is contained in the interval $[0, n]$.

THEOREM 3.5. *Let $0 < p \le 1$. Then, for every entire function f of exponential type less than or equal to n,*

$$\| f^{(\alpha)} \|_{H^p(\mathbb{R})} \le c(\alpha) n^\alpha \| f \|_{H^p(\mathbb{R})}.$$

The proof is analogous. □

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