J. Austral. Math. Soc. (*Series A*) **65** (1999), 354–369

AN ALGEBRAIC APPROACH TO WIGNER'S UNITARY-ANTIUNITARY THEOREM

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(Received 17 February 1998; revised 13 August 1998)

Communicated by P. G. Dodds

Dedicated to Réka Anna

Abstract

We present an operator algebraic approach to Wigner's unitary-antiunitary theorem using some classical results from ring theory. To show how effective this approach is, we prove a generalization of this celebrated theorem for Hilbert modules over matrix algebras. We also present a Wigner-type result for maps on prime C^*-algebras.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 46C05, 46C50; secondary 47D25, 16N60.
Keywords and phrases: Wigner's unitary-antiunitary theorem, Hilbert module, C^*-algebra, prime ring, Jordan homomorphism.

1. Introduction and statement of the results

Wigner's unitary-antiunitary theorem reads as follows. Let H be a complex Hilbert space and let $T : H \to H$ be a surjective map (linearity is not assumed) with the property that

$$|\langle Tx, Ty\rangle| = |\langle x, y\rangle| \qquad (x, y \in H).$$

Then T is of the form

$$Tx = \varphi(x)Ux \qquad (x \in H),$$

This research was supported from the following sources: (1) Hungarian National Foundation for Scientific Research (OTKA), Grant No. T–016846 F–019322, (2) A grant from the Ministry of Education, Hungary, Reg. No. FKFP 0304/1997.

where $U : H \to H$ is either a unitary or an antiunitary operator (that is, U is either an inner product preserving linear bijection or a bijective conjugate-linear map with the property that $\langle Ux, Uy\rangle = \langle y, x\rangle$ for all $x, y \in H$) and $\varphi : H \to \mathbb{C}$ is a so-called phase-function which means that its values are of modulus one. This celebrated result plays a very important role in quantum mechanics and in representation theory in physics.

There are several proofs of this theorem in the literature. See, for example, [LoMe], [Rät], [ShAl], [Uhl] and the references therein. The common characteristic of the arguments presented in those papers is that they manipulate within the Hilbert space which seems to be very natural, of course. In this paper we offer a different approach to Wigner's theorem. Namely, instead of working in H, we push the problem to a certain operator algebra over H and apply some well-known results from ring theory to obtain the desired conclusion. We should remark that in relation to Wigner's theorem, operator algebras appear also in the papers of Uhlhorn [Uhl] and Wright [Wri]. However, in [Uhl] they have nothing to do with the proof of the unitary-antiunitary theorem. Indeed, Uhlhorn presents an argument which can be classified into the first mentioned group of proofs. Moreover, in [Wri] the author uses Gleason's theorem at a crucial point of the proof which is a deep result with long proof. The advantage of our algebraic approach is that in the classical case, our proof is very clear and short, and it uses only a well-known theorem of Herstein on Jordan homomorphisms of rings whose proof needs only few lines of elementary algebraic computation. It is noteworthy that this result of Herstein was known before the first complete proofs of Wigner's theorem appeared. Furthermore, which is more important, our approach makes it possible to generalize Wigner's original theorem for Hilbert modules, that is, for inner product structures where the inner product takes its values in an algebra, not necessarily in the complex field. Considering the previously mentioned proofs, they are based on such characteristic properties of the complex field that one would meet very serious difficulties if one tried to reach our more general result using those methods (in fact, we are convinced that such an approach simply cannot be successful).

We now turn to our results. Let A be a C^*-algebra. Let $\mathscr{H}$ be a left A-module with a map $[\cdot,\cdot] : \mathscr{H} \times \mathscr{H} \to A$ satisfying

(i) $[f+g, h] = [f, h] + [g, h]$
(ii) $[af, g] = a[f, g]$
(iii) $[g, f] = [f, g]^*$
(iv) $[f, f] \geq 0$ and $[f, f] = 0$ if and only if $f = 0$

for every $f, g, h \in \mathscr{H}$ and $a \in A$. If $\mathscr{H}$ is complete with respect to the the norm $f \mapsto \|[f, f]\|^{1/2}$, then we say that $\mathscr{H}$ is a Hilbert A-module with generalized inner product $[\cdot,\cdot]$. This concept is due to Kaplansky [Kap] and in its full generality to Paschke [Pas]. Nowadays, Hilbert modules over C^*-algebras play a very important role for example in the K-theory of C^*-algebras.

There is another concept of Hilbert modules due to Saworotnow [Saw]. These are modules over H^*-algebras. H^*-algebras are common generalizations of L^2-algebras (convolution algebras) of compact groups and Hilbert-Schmidt operator algebras on Hilbert spaces. The only formal difference in the definition is that in the case of Saworotnow's modules, the generalized inner product takes its values in the trace-class of the underlying H^*-algebra and the norm with respect to which we require completeness is $f \mapsto (\mathrm{tr}[f, f])^{1/2}$. Here, tr denotes the trace-functional corresponding to A (see [SaFr]). We should note that Saworotnow originally posed another axiom, namely, a Schwarz-type inequality [Saw, Definition 1]. However, as we proved in [Mol1, Theorem], this axiom is redundant. Saworotnow's modules appear naturally when dealing with multivariate stochastic processes (see [WiMa, Section 5], [Mas]). Moreover, as it turns out from [Cno, Section 3], for example, they have applications in Clifford analysis and hence in some parts of mathematical physics. The theory of these modules is more satisfactory in the sense that many more Hilbert space-like results have counterparts in Hilbert modules over H^*-algebras than in Hilbert modules over C^*-algebras. Note that it seems to be more common to use right modules instead of left ones. Of course, this is not a real difference, only a question of taste.

If $A = M_d(\mathbb{C})$ the algebra of all $d \times d$ complex matrices, then, A being finite dimensional, the norms on A are all equivalent. Therefore, the Hilbert modules over the C^*-algebra $M_d(\mathbb{C})$ are the same as the Hilbert modules over the H^*-algebra $M_d(\mathbb{C})$.

Theorem 1 generalizes the original unitary-antiunitary theorem. As usual, in a C^*-algebra A, $|a|$ denotes the absolute value of the element a which is the unique positive square-root of a^*a. If $\mathscr{H}$ is a Hilbert module, then the linear bijection $U : \mathscr{H} \to \mathscr{H}$ is called A-unitary if $U(af) = aUf$ $(f \in \mathscr{H}, a \in A)$ and $[Uf, Uf'] = [f, f']$ holds true for every $f, f' \in \mathscr{H}$.

THEOREM 1. *Let $\mathscr{H}$ be a Hilbert module over the matrix algebra $A = M_d(\mathbb{C})$ and suppose that there exist vectors $g, h \in \mathscr{H}$ such that $[g, h] = I$. Let $T : \mathscr{H} \to \mathscr{H}$ be a surjective function with the property that*

$$|[Tf, Tf']| = |[f, f']| \qquad (f, f' \in \mathscr{H}). \tag{1}$$

If $d > 1$, then there exist an A-unitary operator $U : \mathscr{H} \to \mathscr{H}$ and a phase-function $\varphi : \mathscr{H} \to \mathbb{C}$ such that

$$Tf = \varphi(f)Uf \qquad (f \in \mathscr{H}).$$

If $d = 1$, then there exist an either unitary or antiunitary operator U on $\mathscr{H}$ and a phase-function $\varphi : \mathscr{H} \to \mathbb{C}$ such that

$$Tf = \varphi(f)Uf \qquad (f \in \mathscr{H}).$$

It seems natural to ask what happens if A is an infinite dimensional algebra. We have the following result for trivial modules over prime C^*-algebras. If $\mathscr{A}$ is a C^*-algebra, then $\mathscr{A}$ is a left module over itself and if we set $[f, g] = fg^*$ $(f, g \in \mathscr{A})$, then $\mathscr{A}$ becomes a Hilbert module over $\mathscr{A}$. This is what we mean when speaking about trivial modules. A ring $\mathscr{R}$ is called prime if for any $a, b \in \mathscr{R}$, the relation $a\mathscr{R}b = \{0\}$ implies that either $a = 0$ or $b = 0$. For example, every algebra of operators which contains the ideal of all finite rank operators is easily seen to be prime. Moreover, von Neumann algebras with trivial centre, that is, factors, are prime C^*-algebras.

THEOREM 2. *Let $\mathscr{A}$ be a prime C^*-algebra with unit and let $\phi : \mathscr{A} \to \mathscr{A}$ be a surjective function such that*

$$|\phi(A)\phi(B)^*| = |AB^*| \qquad (A, B \in \mathscr{A}). \tag{2}$$

Then there exist a unitary element $U \in \mathscr{A}$ and a phase-function $\varphi : \mathscr{A} \to \mathbb{C}$ such that ϕ is of the form

$$\phi(A) = \varphi(A)AU \qquad (A \in \mathscr{A}).$$

This result is in accordance with Theorem 1. In fact, every A-linear operator on the trivial module $\mathscr{A}$ is equal to the operator of right multiplication by an element of $\mathscr{A}$. It is easy to see that if such a map is A-unitary, then the corresponding element of $\mathscr{A}$ is unitary.

Finally, we give a new proof of the real version of Wigner's theorem.

THEOREM 3. *Let H be a real Hilbert space and $T : H \to H$ be a surjective function with the property that*

$$|\langle Tx, Ty\rangle| = |\langle x, y\rangle| \qquad (x, y \in H). \tag{3}$$

Then there exist a unitary operator $U : H \to H$ and a function $\varphi : H \to \{-1, 1\}$ such that T is of the form

$$Tx = \varphi(x)Ux \qquad (x \in H).$$

The proofs of the results are based on the following theorems from ring theory:

- Herstein's homomorphism-antihomomorphism theorem for Jordan homomorphisms which map onto prime rings.
- A result of Martindale on elementary operators on prime rings.
- A theorem of Martindale (or a result of Jacobson and Rickart) on the extendability of Jordan homomorphisms defined on the symmetric elements of a ring with involution.

2. Proofs

As mentioned in the introduction, Saworotnow's modules have many convenient properties which are familiar in the theory of Hilbert spaces. First of all, if $\mathscr{H}$ is a Hilbert module over an H^*-algebra A, then $\mathscr{H}$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle = \mathrm{tr}[\cdot, \cdot]$. If $M \subset \mathscr{H}$ is a closed submodule, then for the closed submodule $M^p = \{f \in \mathscr{H} : [f, g] = 0\,(g \in M)\}$ we obtain $M^p = M^\perp$. So, we have the orthogonal decomposition $\mathscr{H} = M \oplus M^p$ [Saw, Lemma 3]. A linear operator T on $\mathscr{H}$ which is bounded with respect to the Hilbert space norm defined above is called an A-linear operator if $T(af) = aTf$ holds true for every $f \in \mathscr{H}$ and $a \in A$. Every A-linear operator T is adjointable, namely, the adjoint T^* of T in the Hilbert space sense is A-linear and we have $[Tf, g] = [f, T^*g]$ $(f, g \in \mathscr{H})$ [Saw, Theorem 4]. Consequently, the collection of all A-linear operators forms a C^*-subalgebra of the full operator algebra on the Hilbert space $\mathscr{H}$.

For the proof of our Theorem 1 we need the following lemma. In the case of a Hilbert module $\mathscr{H}$ over an H^*-algebra, the natural equivalent of the Hilbert base is the so-called modular base [Mol2]. A family $\{f_\alpha\}_\alpha \subset \mathscr{H}$ is said to be modular orthonormal if

(a) $[f_\alpha, f_\beta] = 0$ if $\alpha \neq \beta$,
(b) $[f_\alpha, f_\alpha]$ is a minimal projection in A for every α.

A maximal modular orthonormal family of vectors in $\mathscr{H}$ is called a modular base. The common cardinality of modular bases in $\mathscr{H}$ is called the modular dimension of $\mathscr{H}$ (see [Mol2, Theorem 2]).

LEMMA 1. *Let $\mathscr{H}$ be a Hilbert A-module over the matrix algebra $A = M_d(\mathbb{C})$. If $M \subset \mathscr{H}$ is a submodule which is generated by finitely many vectors, then M has finite modular dimension.*

PROOF. Observe that since A is finite dimensional, the submodule generated by finitely many vectors has finite linear dimension. Therefore, every such submodule is closed. Let M be generated by the vectors $f_1, \dots, f_n$. Consider the submodule $M_1 = Af_1 \subset M$. By orthogonal decomposition we can write $f_2 = g_2 + h_2$, where $h_2 \in M_1$, $g_2 \in M \cap M_1^p$. Clearly, $M_2 = Ag_2 \subset M_1^p$ and we have $f_1, f_2 \in M_1 + M_2 \subset M$. Next, let $f_3 = g_3 + h_3$, where $h_3 \in M_1 + M_2$ and $g_3 \in M \cap (M_1 + M_2)^p$. Let $M_3 = Ag_3$. We have $f_1, f_2, f_3 \in M_1 + M_2 + M_3 \subset M$. Continuing the process we obtain vectors $g_1, g_2, \dots, g_n$ with $[g_i, g_j] = 0$ $(i \neq j)$ for which $f_1, \dots, f_n$ is included in the submodule generated by $g_1, \dots, g_n$. Consequently, M is generated by the g_k's.

Let $g \in \mathscr{H}$ be a nonzero vector. Write $[g, g] = \sum_n \lambda_n^2 e_n$, where the e_k's are pairwise orthogonal minimal projections. Let $h_k = (1/\lambda_k)e_k g$. Apparently, we have $[h_i, h_j] = 0$ $(i \neq j)$ and $[h_k, h_k]$ is a minimal projection. We assert that $\sum_k \lambda_k e_k h_k = g$. This can be verified by taking the generalized inner product of both sides of this equation with g and then with any vector $f \in \mathscr{H}$ for which $[f, g] = 0$. Collecting the h's corresponding to the generating vectors $g_1, \dots, g_n$ of M, by [Mol2, Theorem 1] we obtain a finite modular base in M. $\square$

REMARK 1. The previous lemma tells us that, under the above assumption on $\mathscr{H}$, a submodule of $\mathscr{H}$ has finite modular dimension if and only if it has a finite linear dimension.

To emphasize how different the behaviour of Hilbert modules can be from that of Hilbert spaces, we note that in general the statement of the previous lemma does not hold true for Hilbert modules over infinite dimensional H^*-algebras.

In what follows we define operators which are the natural equivalent of the finite rank operators in the case of Hilbert spaces. If $f, g \in \mathscr{H}$, then let $f \odot g$ denote the A-linear operator defined by

$$(f \odot g)h = [h, g]f \qquad (h \in \mathscr{H}).$$

It is easy to see that for every A-linear operator S we have

$$S(f \odot g) = (Sf) \odot g, \qquad (f \odot g)S = f \odot (S^* g) \tag{4}$$

and

$$(f \odot g)(f' \odot g') = ([f', g]f) \odot g' = f \odot ([g, f']g'). \tag{5}$$

Define

$$\mathscr{F}(\mathscr{H}) = \left\{ \sum_{k=1}^{n} f_k \odot g_k : f_k, g_k \in \mathscr{H} \ (k = 1, \dots, n), \ n \in \mathbb{N} \right\}$$

which is a *-ideal of the C^*-algebra of all A-linear operators. We note that if $\mathscr{H}$ is a Hilbert module over $M_d(\mathbb{C})$, then the range of every element of $\mathscr{F}(\mathscr{H})$ has finite linear dimension, but there can be finite rank operators on the Hilbert space $\mathscr{H}$ which do not belong to $\mathscr{F}(\mathscr{H})$. In general, if the underlying H^*-algebra is infinite dimensional, then these two classes of operators have nothing to do with each other.

The following lemma is a spectral theorem for the self-adjoint elements of $\mathscr{F}(\mathscr{H})$.

LEMMA 2. *Let $\mathscr{H}$ be a Hilbert module over the matrix algebra $A = M_d(\mathbb{C})$. If $S \in \mathscr{F}(\mathscr{H})$ is a self-adjoint operator, then S can be written in the form*

$$S = \sum_{k=1}^{n} \lambda_k f_k \odot f_k$$

where $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ and $\{f_1, \dots, f_n\} \subset \mathscr{H}$ is modular orthonormal.

PROOF. Let $S \in \mathscr{F}(\mathscr{H})$ be a self-adjoint operator. Since the range of S has finite linear dimension, S can be written in the form

$$S = \sum_k \lambda_k E_k,$$

where the λ_k's are the pairwise different nonzero eigenvalues of S and the E_k's are the corresponding spectral projections. Since S is A-linear, its eigensubspaces are submodules. Hence, every spectral projection is A-linear with range included in the range of S. Lemma 1 yields that the range of E_k has finite modular dimension. Choose a modular base in the range of every E_k. Using the analog of the Fourier expansion given in [Mol2, Theorem 1, (iv)] we easily conclude that S can be written in the desired form. □

Now, we are in a position to prove our first theorem. For the proof we need the concept of Jordan homomorphisms. A linear map ϕ between algebras $\mathscr{A}$ and $\mathscr{B}$ is said to be a Jordan homomorphism if it satisfies

$$\phi(x)^2 = \phi(x^2) \qquad (x \in \mathscr{A}),$$

or equivalently

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x) \qquad (x, y \in \mathscr{A}).$$

PROOF OF THEOREM 1. We define a linear transformation ψ on the set of all self-adjoint elements of $\mathscr{F}(\mathscr{H})$ as follows. For any $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ and $\{f_1, \dots, f_n\} \subset \mathscr{H}$ (we do not require modular orthonormality) if

$$S = \sum_k \lambda_k f_k \odot f_k, \tag{6}$$

then let

$$\psi(S) = \sum_k \lambda_k T f_k \odot T f_k.$$

To see that ψ is well-defined, let $\mu_l \in \mathbb{R}$ and $g_l \in \mathscr{H}$ be such that

$$\sum_k \lambda_k f_k \odot f_k = \sum_l \mu_l g_l \odot g_l.$$

We compute

$$\begin{aligned}
\left[\left(\sum_k \lambda_k T f_k \odot T f_k\right) Th, Th\right] &= \sum_k \lambda_k [Th, T f_k][T f_k, Th] \\
&= \sum_k \lambda_k [h, f_k][f_k, h] = \left[\left(\sum_k \lambda_k f_k \odot f_k\right) h, h\right] \\
&= \left[\left(\sum_l \mu_l g_l \odot g_l\right) h, h\right] = \sum_l \mu_l [h, g_l][g_l, h] \\
&= \sum_l \mu_l [Th, Tg_l][Tg_l, Th] = \left[\left(\sum_l \mu_l Tg_l \odot Tg_l\right) Th, Th\right].
\end{aligned}$$

Since T is surjective, we obtain that ψ is well-defined. Due to the fact that in the form (6) of S we have not required anything from the vectors f_k, we obtain readily that ψ is additive and real linear.

We next show that ψ is a Jordan homomorphism. Let

$$S = \sum_k \lambda_k f_k \odot f_k$$

where $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ and $\{f_1, \dots, f_n\} \subset \mathscr{H}$ is modular orthonormal. If $\{f, g\} \subset \mathscr{H}$ is modular orthonormal, then according to (5) we have

$$f \odot f \cdot g \odot g = 0$$

and

$$f \odot f \cdot f \odot f = ([f, f] f) \odot f = f \odot f$$

where we have used the equality $[f, f] f = f$ (see [Mol2, Lemma 1]). Therefore, we have $S^2 = \sum_k \lambda_k^2 f_k \odot f_k$. Since $\{T f_1, \dots, T f_n\}$ is modular orthonormal, we have $\psi(S)^2 = \sum_k \lambda_k^2 T f_k \odot T f_k$. This results in

$$\psi(S)^2 = \psi(S^2).$$

Consequently, ψ is a Jordan homomorphism, more precisely, a Jordan automorphism of the self-adjoint elements of $\mathscr{F}(\mathscr{H})$. Linearizing the equality above, that is, replacing S by $S + R$ we deduce

$$\psi(S)\psi(R) + \psi(R)\psi(S) = \psi(SR + RS)$$

for every self-adjoint $S, R \in \mathscr{F}(\mathscr{H})$. It is now easy to check that the map $\Psi : \mathscr{F}(\mathscr{H}) \to \mathscr{F}(\mathscr{H})$ defined by

$$\Psi(S + iR) = \psi(S) + i\psi(R)$$

for every self-adjoint $S, R \in \mathscr{F}(\mathscr{H})$ is a Jordan *-automorphism of $\mathscr{F}(\mathscr{H})$ which extends ψ.

We claim that $\mathscr{F}(\mathscr{H})$ is a prime ring. Let S, R be A-linear operators such that $S(f \odot g)R = 0$ holds true for every $f, g \in \mathscr{H}$. For an arbitrary $a \in A$ we infer

$$[Rh, g]a[Sf, h'] = [Rh, g][S(af), h'] = [S(af \odot g)Rh, h'] = 0 \quad (h, h' \in \mathscr{H}).$$

Since A is clearly a prime ring, we obtain that for every f, g, h, h' we have either $[Rh, g] = 0$ or $[Sf, h'] = 0$. This implies that either $S = 0$ or $R = 0$ holds true verifying the primeness of $\mathscr{F}(\mathscr{H})$.

A well-known theorem of Herstein [Her1] says that every Jordan homomorphism onto a prime algebra is either a homomorphism or an antihomomorphism. Accordingly, Ψ is either a *-automorphism or a *-antiautomorphism of $\mathscr{F}(\mathscr{H})$. Suppose first that it is a *-automorphism. Let $g, h \in \mathscr{H}$ be fixed vectors with the property that $[g, h] = I$. Define a linear operator $U : \mathscr{H} \to \mathscr{H}$ by

$$Uf = \Psi(f \odot g)Th \qquad (f \in \mathscr{H}).$$

For any $R \in \mathscr{F}(\mathscr{H})$ we have

$$URf = \Psi(Rf \odot g)Th = \Psi(R)\Psi(f \odot g)Th = \Psi(R)Uf \qquad (f \in \mathscr{H}). \tag{7}$$

Using (5) and (1) we compute

$$\begin{aligned}
[Uf, Uf] &= [\Psi(g \odot f)\Psi(f \odot g)Th, Th] = [\Psi(g \odot f \cdot f \odot g)Th, Th] \\
&= [\Psi(\sqrt{[f, f]}g \odot \sqrt{[f, f]}g)Th, Th] = [T(\sqrt{[f, f]}g) \odot T(\sqrt{[f, f]}g)Th, Th] \\
&= [Th, T(\sqrt{[f, f]}g)][T(\sqrt{[f, f]}g), Th] = [h, \sqrt{[f, f]}g][\sqrt{[f, f]}g, h] \\
&= [h, g][f, f][g, h] = [f, f].
\end{aligned}$$

Clearly, U is injective. Moreover, just as in the case of Hilbert spaces, by polarization we obtain

$$[Uf, Uf'] = [f, f'] \qquad (f, f' \in \mathscr{H}). \tag{8}$$

To show the surjectivity of U we compute

$$\begin{aligned}
URg &= \Psi(R)Ug = \Psi(R)\Psi(g \odot g)Th = \Psi(R)(Tg \odot Tg)Th \\
&= \Psi(R)([Th, Tg]Tg) = [Th, Tg]\Psi(R)Tg.
\end{aligned}$$

For an arbitrary $f \in \mathscr{H}$ we have $\Psi(R) = f \odot Th$ for some $R \in \mathscr{F}(\mathscr{H})$. Thus the range of U contains the vector

$$[Th, Tg](f \odot Th)(Tg) = [Th, Tg][Tg, Th]f = [h, g][g, h]f = f,$$

verifying the surjectivity of U. Now, by (8) it follows that U is A-linear and hence an A-unitary operator.

From (7) we get $\Psi(R) = URU^*$ $(R \in \mathscr{F}(\mathscr{H}))$. Therefore, for every $f \in \mathscr{H}$ we obtain

$$Tf \odot Tf = \Psi(f \odot f) = U(f \odot f)U^* = Uf \odot Uf.$$

In view of (1), this gives us that

$$\begin{aligned}[][f', f][f, f'] &= [Tf', Tf][Tf, Tf'] = [(Tf \odot Tf)Tf', Tf'] \\ &= [(Uf \odot Uf)Tf', Tf'] = [Tf', Uf][Uf, Tf'] \\ &= [U^*Tf', f][f, U^*Tf']\end{aligned}$$

holds true for every $f, f' \in \mathscr{H}$. Replacing f by xf $(x \in A)$, we deduce

$$[f', f]x^*x[f, f'] = [U^*Tf', f]x^*x[f, U^*Tf'].$$

Since every $x \in A$ is a linear combination of positive elements, we have

$$[f', f]x[f, f'] = [U^*Tf', f]x[f, U^*Tf'] \qquad (x \in A). \tag{9}$$

According to a result of Martindale [Mar2] (see [Her2, Lemma 1.3.2]), if an elementary operator $x \mapsto \sum_{k=1}^{n} a_k x b_k$ defined on a prime ring $\mathscr{R}$ is identically 0, then $a_1, \ldots, a_n \in \mathscr{R}$ are linearly dependent over the extended centroid of $\mathscr{R}$ and the same is true for $b_1, \ldots, b_n \in \mathscr{R}$. By the remark after [Mat, Proposition 2.5], the extended centroid of a prime C^*-algebra is just $\mathbb{C}$ (this remarkable fact will be used also in the proof of our Theorem 2). So, from (9) we get that for every $f, f' \in \mathscr{H}$ the elements $[f, f']$ and $[f, U^*Tf']$ of A are linearly dependent. Fix $f' \in \mathscr{H}$. We know that the linear operators $f \mapsto [f, f']$ and $f \mapsto [f, U^*Tf']$ are locally linearly dependent. It is elementary linear algebra to verify that in this case these operators are (globally) linearly dependent. Hence, we conclude that for every $f' \in \mathscr{H}$ there is a scalar $\varphi(f')$ such that $\varphi(f')f' = U^*Tf'$. It follows that

$$Tf = \varphi(f)Uf \qquad (f \in \mathscr{H}).$$

Since

$$\|Tf\|^2 = \operatorname{tr}[Tf, Tf] = \operatorname{tr}[f, f] = \operatorname{tr}[Uf, Uf] = \|Uf\|^2 \qquad (f \in \mathscr{H}),$$

we obtain that φ is a phase-function.

It remains to consider the case when Ψ is *-antiautomorphism. Just as above, let $g, h \in \mathscr{H}$ be fixed such that $[g, h] = I$. Define $U : \mathscr{H} \to \mathscr{H}$ by

$$Uf = \Psi(g \odot f)Th \qquad (f \in \mathscr{H}).$$

Clearly, U is a conjugate-linear operator. Similarly to (7), it is easy to verify that

$$URf = \Psi(R)^* Uf \tag{10}$$

holds true for every $R \in \mathcal{F}(\mathcal{H})$ and $f \in \mathcal{H}$. Moreover, just as in the case when Ψ is a *-automorphism, we obtain

$$[Uf, Uf] = [f, f] \qquad (f \in \mathcal{H}). \tag{11}$$

In particular, U is injective. By (10), using an argument similar to what we have followed in the first case, one can check that U is surjective. By the conjugate-linearity of U, (11) yields

$$[Uf, Uf'] = [f', f] \qquad (f, f' \in \mathcal{H}).$$

Let $f \in \mathcal{H}$ and define $S = f \odot f$. From (10) we obtain

$$\begin{aligned}[\zeta, f][f, \xi] &= [(f \odot f)\zeta, \xi] = [\zeta, S\xi] = [U(S\xi), U\zeta] \\ &= [\Psi(S)^*(U\xi), U\zeta] = [(Tf \odot Tf)(U\xi), U\zeta] = [U\xi, Tf][Tf, U\zeta]\end{aligned}$$

for every $\zeta, \xi \in \mathcal{H}$. This gives us that

$$[U^{-1}T\xi, f][f, U^{-1}T\xi] = [T\xi, Tf][Tf, T\xi] = [\xi, f][f, \xi]$$

holds true for every $f, \xi \in \mathcal{H}$. Fixing ξ, just as in the case when Ψ is an automorphism, we obtain that $[f, \xi]$ and $[f, U^{-1}T\xi]$ are linearly dependent for every f. Therefore, ξ and $U^{-1}T\xi$ are linearly dependent for every $\xi \in \mathcal{H}$. This shows that there exists a phase-function $\varphi : \mathcal{H} \to \mathbb{C}$ such that

$$Tf = \varphi(f)Uf \qquad (f \in \mathcal{H}). \tag{12}$$

The proof is now complete in the case when $d = 1$.

Suppose that $d > 1$. In the antiautomorphic case, by (12) we have

$$|[f, f']| = |[Tf, Tf']| = |[Uf, Uf']| = |[f', f]| \qquad (f, f' \in \mathcal{H}).$$

Since there are vectors $g, h \in \mathcal{H}$ such that $[g, h] = I$, it follows that $|a| = |a^*|$ holds true for every $a \in A = M_d(\mathbb{C})$. As $d > 1$, it is an obvious contradiction, so this case cannot arise. □

REMARK 2. Observe that if $n = 1$, that is, when we have the classical situation of Hilbert spaces, our proof is much shorter (see [Mol3, Theorem 1] where this case was treated) and uses only Herstein's homomorphism-antihomomorphism theorem whose proof needs only few lines of algebraic computation (see, for example, [Bre, Theorem 2.1] or [Pal, 6.3.2 Lemma, 6.3.6 Lemma and 6.3.7 Theorem]).

From the proof of Theorem 1 it should be clear why we have considered modules over matrix algebras. Namely, by the structure theorem of H^*-algebras due to Ambrose [Amb], the full matrix algebras are the only unital prime H^*-algebras.

One may ask the meaning of the existence of two vectors $g, h \in \mathscr{H}$ with the property $[g, h] = I$ which appeared in the formulation of the theorem above. We claim that this is equivalent to the requirement that the modular dimension of $\mathscr{H}$ is not less than d. To see this, let $\{f_1, \ldots, f_d\} \subset \mathscr{H}$ be modular orthonormal. Choose appropriate matrices $a_i \in M_d(\mathbb{C})$ such that for the vectors $g_i = a_i f_i$ we have $[g_i, g_i] = a_i[f_i, f_i]a_i^* = e_{ii}$ $(i = 1, \ldots, d)$, the standard matrix units. It follows that $[g_1 + \cdots + g_d, g_1 + \cdots + g_d] = I$. Now, let the modular dimension of $\mathscr{H}$ be less than d and choose a modular base $\{f_1, \ldots, f_n\} \subset \mathscr{H}$, where $n < d$. By [Mol2, Theorem 1, (v)] and [Mol2, Lemma 1] we have

$$[g, h] = \sum_{k=1}^{n} [g, f_k][f_k, h] = \sum_{k=1}^{n} [g, f_k][f_k, f_k][f_k, h].$$

Since $[f_k, f_k]$ is a rank-one projection, we obtain that the rank of $[g, h]$ is not greater than n. This shows that $[g, h] \neq I$ for every $g, h \in \mathscr{H}$. It would be interesting to investigate Wigner's theorem also in these low-dimensional cases.

PROOF OF THEOREM 2. Let $A, B \in \mathscr{A}$ be arbitrary. Define

$$\psi(A^*A - B^*B) = \phi(A)^*\phi(A) - \phi(B)^*\phi(B).$$

To see that ψ is well-defined, let $A', B' \in \mathscr{A}$ be such that

$$A^*A - B^*B = A'^*A' - B'^*B'.$$

For every $S \in \mathscr{A}$ have

$$SA^*AS^* - SB^*BS^* = SA'^*A'S^* - SB'^*B'S^*$$

and by (2) we deduce

$$\begin{aligned}\phi(S)\phi(A)^*\phi(A)\phi(S)^* - \phi(S)\phi(B)^*\phi(B)\phi(S)^* \\ = \phi(S)\phi(A')^*\phi(A')\phi(S)^* - \phi(S)\phi(B')^*\phi(B')\phi(S)^*.\end{aligned}$$

By the surjectivity of ϕ there exists an $S \in \mathscr{A}$ for which $\phi(S) = I$. We infer

$$\phi(A)^*\phi(A) - \phi(B)^*\phi(B) = \phi(A')^*\phi(A') - \phi(B')^*\phi(B').$$

Therefore, ψ is a well-defined map on the self-adjoint elements of $\mathscr{A}$. Using an argument very similar to the one we have just applied, one can prove that ψ is additive. We claim that $\psi(I) = I$. Let $\phi(S) = I$. We have $I = \phi(S)^*\phi(S) = \psi(S^*S)$. Since ψ is positivity preserving and hence monotone, by the inequality $S^*S \le \|S\|^2 I \le nI$, which holds true for some $n \in \mathbb{N}$, we infer that

$$I = \psi(S^*S) \le n\psi(I). \tag{13}$$

On the other hand, $\psi(I)$ is a projection. In fact, we have $\psi(I) = \phi(I)^*\phi(I)$. From (2) it follows that $\psi(I)^2 = \psi(I)$. By the spectral mapping theorem this means that the spectrum of $\psi(I)$ is included in $\{0, 1\}$. Since $\psi(I)$ is self-adjoint, we obtain that $\psi(I)$ is a projection. From (13) we now get $\psi(I) = I$. If $U = \phi(I)$, then we have $U^*U = \psi(I) = I$. On the other hand, by (2) it follows that $UU^* = \phi(I)\phi(I)^* = II^* = I$. Consequently, $U \in \mathscr{A}$ is unitary. From (2) we deduce that

$$AT^*TA^* = \phi(A)\phi(T)^*\phi(T)\phi(A)^* = \phi(A)\psi(T^*T)\phi(A)^*$$

holds true for every $A, T \in \mathscr{A}$. Choosing $A = I$, we obtain $\psi(T^*T) = U^*(T^*T)U$. Therefore, we get

$$AT^*TA^* = (\phi(A)U^*)T^*T(\phi(A)U^*)^*.$$

Since this equation holds true for every $T \in \mathscr{A}$, it follows that

$$AXA^* = (\phi(A)U^*)X(\phi(A)U^*)^* \qquad (X \in \mathscr{A}).$$

By the primeness of $\mathscr{A}$, using [Her2, Lemma 1.3.2] and the remark after [Mat, Proposition 2.5] just as in the proof of Theorem 1, it follows that for every $A \in \mathscr{A}$ the elements A and $\phi(A)U^*$ are linearly dependent. Consequently, there exists a scalar valued function $\varphi : \mathscr{A} \to \mathbb{C}$ such that

$$\varphi(A)AU = \phi(A) \qquad (A \in \mathscr{A}).$$

This relation yields that

$$|\varphi(A)|^2\|A\|^2 = \|\phi(A)\|^2 = \|\phi(A)\phi(A)^*\| = \|AA^*\| = \|A\|^2 \quad (A \in \mathscr{A})$$

which implies $|\varphi(A)| = 1$ $(0 \ne A \in \mathscr{A})$. □

PROOF OF THEOREM 3. If $x, y \in H$, then let $x \otimes y$ be the rank-one operator defined by $(x \otimes y)z = \langle z, y\rangle x$ $(z \in H)$. By the real version of the spectral theorem, every symmetric (that is, real self-adjoint) finite rank operator S can be written in the form

$$S = \sum_{k=1}^{n} \lambda_k x_k \otimes x_k,$$

where $\lambda_k \in \mathbb{R}$ and $x_k \in H$. Similarly to the proof of Theorem 1, we define $\psi(S)$ by the formula

$$\psi(S) = \sum_{k=1}^{n} \lambda_k Tx_k \otimes Tx_k.$$

Repeating the argument in the corresponding part of the proof of Theorem 1, we see that ψ is a Jordan automorphism of the symmetric elements of the ring $F(H)$ of all finite rank operators on H. In what follows suppose that $\dim H \geq 2$. In fact, if H is one-dimensional, then the statement of the theorem is trivial.

Consider the unitalized algebra $F(H) \oplus \mathbb{R}I$ (of course, we have to add the identity only in the infinite dimensional case). Defining $\psi(I) = I$, we can extend ψ to the set of all symmetric elements of the enlarged algebra in an obvious way. Now we are in a position to apply two general algebraic results of Martindale on the extension of Jordan homomorphisms of the symmetric elements of rings with involution [Mar1]. To be precise, in [Mar1] Jordan homomorphism means an additive map ϕ which, besides $\phi(s)^2 = \phi(s^2)$, satisfies $\phi(sts) = \phi(s)\phi(t)\phi(s)$ as well. But if the ring in question is 2-torsion free (in particular, if it is an algebra), this second equality follows from the first one (see, for example, the proof of [Pal, 6.3.2 Lemma]). The statements [Mar1, Theorem 1] in the case $\dim H \geq 3$ and [Mar1, Theorem 2] when $\dim H = 2$ imply that ψ can be extended uniquely to an associative homomorphism of $F(H) \oplus \mathbb{R}I$ into itself. To be honest, since the results of Martindale concern rings and hence linearity does not appear, we could guarantee only the additivity of the extension of ψ. However, the construction in [Mar1] clearly shows that in the case of algebras, linear Jordan homomorphisms have linear extensions. By the uniqueness of the extension it is apparent that the extension is *-preserving. Next, observe that our extension maps $F(H)$ into itself. Thus we have an associative *-homomorphism Ψ of $F(H)$ into itself which extends ψ. It is easy to see that Ψ is a bijection. Indeed, for arbitrary nonzero vectors $x, y \in H$ pick a vector $z \in H$ with $\langle x, z\rangle, \langle y, z\rangle \neq 0$. Plainly, $x \otimes y$ is a nonzero scalar multiple of the operator $x \otimes x \cdot z \otimes z \cdot y \otimes y$. Since our ψ is a bijection from the set of symmetric elements of $F(H)$ onto itself and Ψ is an (associative) homomorphism, we obtain that every rank-one operator is in the range of Ψ. This proves the surjectivity of Ψ. The injectivity follows from the simplicity of the algebra $F(H)$.

The form of *-automorphisms of subalgebras of the full operator algebra on H containing $F(H)$ is well-known. It follows easily from [Che, 3.2. Corollary], for example, that there is a unitary operator U on H for which

$$\Psi(A) = UAU^* \qquad (A \in F(H)).$$

This gives us that

$$Tx \otimes Tx = \Psi(x \otimes x) = U(x \otimes x)U^* = (Ux) \otimes (Ux) \qquad (x \in H).$$

This implies that Tx is a scalar multiple of Ux, and the scalar must be of modulus one. The proof is now complete. $\square$

REMARK 3. Observe that in the case when $n \geq 3$ we could have used a theorem of Jacobson and Rickart [JaRi, Theorem 5]. Nevertheless, we referred to Martindale's paper since it covers the two-dimensional case as well.

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