

TIME-ISOLATED SINGULARITIES OF TEMPERATURES

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Abstract

We study singularities of solutions of the heat equation, that are not necessarily isolated but occur only in a single characteristic hyperplane. We prove a decomposition theorem for certain solutions on $D_+ = D \cap (\mathbf{R}^n \times]0, \infty[)$, for a suitable open set D , with singularities at a compact subset K of $\mathbf{R}^n \times \{0\}$, in terms of Gauss-Weierstrass integrals. We use this to prove a representation theorem for certain solutions on D_+ , with singularities at K , as the sums of potentials and Dirichlet solutions. We also give conditions under which K is removable for solutions on $D \setminus K$.

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1. Introduction

Let D be an open subset of $\mathbf{R}^{n+1} = \{(x, t) : x \in \mathbf{R}^n, t \in \mathbf{R}\}$, let $D(0) = \{x \in \mathbf{R}^n : (x, 0) \in D\} \neq \emptyset$, let $D_+ = D \cap (\mathbf{R}^n \times]0, \infty[)$ and let $H^\Delta(D)$ be the family of all temperatures on D that can be written as a difference of nonnegative temperatures. The central result of this paper, Theorem 2, gives conditions under which an element u of $H^\Delta(D_+)$ can be written in the form $u = W\mu + W\psi + w$, where μ is a signed measure supported in a compact subset C of $D(0)$, ψ is a locally integrable function on $D(0)$ such that $W|\psi| < \infty$ on D_+ , and w is a temperature on D_+ that can be extended by zero to a temperature on D . Here

$$W\mu(x, t) = \int_{\text{supp } \mu} W(x - y, t) d\mu(y)$$

and

$$W\psi(x, t) = \int_{\text{supp } \psi} W(x - y, t)\psi(y)dy,$$

with $W(x, t) = (4\pi t)^{-n/2} \exp(-\|x\|^2/4t)$ for all $(x, t) \in \mathbf{R}_+^{n+1}$. Because w tends to zero at $D(0) \times \{0\}$, this decomposition enables us to use theorems on Gauss-Weierstrass integrals to prove results about temperatures in any $H^\Delta(D_+)$.

In Section 3, we use the decomposition theorem to prove a representation theorem, which extends one established by Aronson [2] for solutions of a wide class of parabolic partial differential equations on $B(0, \rho) \times]0, T[$ with singularities at $(0, 0)$. Working only with temperatures, we are able to considerably weaken the constraints on the solutions, replace the circular cylinder by an arbitrary D_+ , and replace the point of singularity by an arbitrary compact subset K of $D(0) \times \{0\}$. A representation of the form $u = G_D\mu + h$ is obtained, where $G_D\mu$ is the potential on D of a signed measure supported in K , and h is a Dirichlet solution on D_+ .

In Section 4, we consider temperatures u on $D \setminus (C \times \{0\})$ for an arbitrary compact subset C of $D(0)$, and give a mild constraint which ensures that they can be written as the sum of a temperature on D and the potential of a signed measure supported in $C \times \{0\}$. The idea here is that, because the restriction of u to $D \setminus \overline{D}_+$ has a continuous extension to u^* (say) on $D \setminus D_+$, we can take $\psi = u^*(\cdot, 0)$ in the decomposition theorem, so that $W\psi + w$ can be extended to a temperature on D . Given this result, known conditions which imply that μ is null can be converted into conditions for $C \times \{0\}$ to be removable.

Many other papers have been written about removable singularities, including [1, 5–7]. Isolated singularities of nonnegative temperatures have been characterized by Widder [16, p. 119], and those of arbitrary temperatures by Chung and Kim [3].

2. The decomposition theorem

If μ and ψ are, respectively, a measure and a function defined on a subset of \mathbf{R}^n , they are assumed to be extended by zero to the whole space. Their restrictions to a set A are denoted by μ_A and ψ_A .

A temperature u on D_+ is called *initially zero* if $u(x, t) \rightarrow 0$ as $(x, t) \rightarrow (y, 0+)$ for all $y \in D(0)$.

A family \mathcal{F} of closed balls is called an *abundant Vitali covering* of \mathbf{R}^n if, given $x \in \mathbf{R}^n$ and $\epsilon > 0$, \mathcal{F} contains uncountably many balls with centre x and radius less than ϵ . See [13] for a discussion.

The proof of the decomposition theorem requires a preliminary theorem.

THEOREM 1. *Suppose that $u = W\mu + v$ on D_+ , where v is an initially zero*

temperature, and μ is a signed measure concentrated on $D(0)$ such that $W|\mu| < \infty$ on D_+ . Let \mathcal{F} be an abundant Vitali covering of \mathbf{R}^n . If there is a signed measure ν concentrated on $D(0)$ such that

$$(1) \quad \lim_{t \rightarrow 0^+} \int_{A \cap V} u(x, t) dx = \nu(A \cap V)$$

whenever $A, V \in \mathcal{F}$, $V \subseteq D(0)$, and $A \cap V \neq \emptyset$, then $\mu = \nu$.

PROOF. By [13, Theorem 7.3(i)], there is an abundant Vitali covering $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $|\mu|(\partial A) = 0$ for all $A \in \mathcal{F}_0$. Given $V \in \mathcal{F}_0$ such that $V \subseteq D(0)$, put $w_V = W\mu_V$ and $w_{D \setminus V} = W\mu_{D(0) \setminus V}$. Then $w_V = u - v - w_{D \setminus V}$ on D_+ .

If $A \in \mathcal{F}_0$ and $A \cap V \neq \emptyset$, then $A \cap V$ is a compact subset of $D(0)$, so that

$$(2) \quad \lim_{t \rightarrow 0^+} \int_{A \cap V} v(x, t) dx = 0.$$

Furthermore, because the boundaries of $A \cap V$ and $A \setminus V$ are both μ -null, it follows from [13, Theorem 7.2(i)] that

$$(3) \quad \lim_{t \rightarrow 0^+} \int_{A \cap V} w_{D \setminus V}(x, t) dx = 0 = \lim_{t \rightarrow 0^+} \int_{A \setminus V} w_V(x, t) dx.$$

Combining (1), (2) and (3), we obtain

$$\lim_{t \rightarrow 0^+} \int_A w_V(x, t) dx = \lim_{t \rightarrow 0^+} \int_{A \cap V} w_V(x, t) = \nu_V(A).$$

On the other hand, if $A \in \mathcal{F}_0$ and $A \cap V = \emptyset$, then it follows from [13, Theorem 7.2(i)] that

$$\lim_{t \rightarrow 0^+} \int_A w_V(x, t) dx = 0 = \nu_V(A).$$

Therefore $\mu_V = \nu_V$, by [13, Theorem 7.3(ii)].

Given any open subset U of $D(0)$, choose a sequence of sets $\{V_k\}$ in \mathcal{F}_0 with union U , and put $X_1 = V_1$, $X_j = V_j \setminus \bigcup_{k=1}^{j-1} V_k$ for all $j \geq 2$. Then, by the above,

$$\mu(U) = \sum_{j=1}^{\infty} \mu(X_j) = \sum_{j=1}^{\infty} \mu_{V_j}(X_j) = \sum_{j=1}^{\infty} \nu_{V_j}(X_j) = \nu(U).$$

The result now follows from the regularity of Radon measures. \square

NOTE. If, in Theorem 1, $u(x, 0+)$ is finite whenever $x \in D(0)$ and the limit exists, then the same is true of $W\mu(x, 0+)$, and the two are equal. Therefore $d\mu(x) = u(x, 0+)dx$, by [12, Theorem 1], so that $u(\cdot, 0+)$ is locally integrable, and

$$\lim_{t \rightarrow 0+} \int_B u(x, t)dx = \int_B u(x, 0+)dx$$

for each bounded Borel subset B of $D(0)$ such that $m_n(\partial B) = 0$, by [13, Theorem 7.2].

THEOREM 2. *Suppose that $u \in H^\Delta(D_+)$, that C is a compact subset of $D(0)$, that \mathcal{F} is an abundant Vitali covering of \mathbf{R}^n , and that ψ is a locally m_n -integrable function on $D(0)$ such that $W|\psi| < \infty$ on D_+ . If*

$$\lim_{t \rightarrow 0+} \int_{A \cap V} u(x, t)dx = \int_{A \cap V} \psi(x)dx$$

whenever $A, V \in \mathcal{F}$, $V \subseteq D(0) \setminus C$, and $A \cap V \neq \emptyset$, then there exist a unique signed measure μ , supported in C and with finite total variation, and a unique initially zero temperature w on D_+ , such that $u = W\mu + W\psi + w$ on D_+ .

PROOF. By [15, Theorem 1], there is a unique signed measure ν on $D(0)$ with the following property. Given any bounded open set E such that $\overline{E} \subseteq D$ and $E(0) \neq \emptyset$, there is a unique initially zero temperature v on E_+ such that $u = W\nu_E + v$ on E_+ .

Choose E such that $C \subseteq E(0)$. Applying Theorem 1 on $E \setminus (C \times \{0\})$, we obtain $d\nu_{E \setminus C}(y) = \psi_{E \setminus C}(y)dy$. Since C is compact $|\nu|(C) < \infty$, so that $W|\nu_E| \leq W|\nu_C| + W|\psi| < \infty$ on D_+ . It follows that $W|\nu| < \infty$ on D_+ , so that there is a unique initially zero temperature w on D_+ such that $u = W\nu + w$ on D_+ , by [15, Theorem 1]. Putting $d\mu(y) = d\nu_C(y) - \psi_C(y)dy$, we obtain

$$u = W\nu + w = W\nu_C + W\psi_{D \setminus C} + w = W\mu + W\psi + w$$

on D_+ , as asserted. □

REMARK. The measure ν , associated with $u \in H^\Delta(D_+)$ by [15, Theorem 1] and described in the first paragraph of the above proof, is called the *initial measure* of u .

If the initial measure of u is absolutely continuous with respect to m_n , then the following corollary may be easier to use than the theorem. Note that, for any $u \in H^\Delta(D_+)$, the limit $u(x, 0+)$ exists and is finite for m_n -almost every $x \in D(0)$, by [15, Theorem 2].

COROLLARY. *Suppose that $u \in H^\Delta(D_+)$, that C is a compact subset of $D(0)$, that $u(x, 0+)$ is finite whenever $x \in D(0) \setminus C$ and the limit exists, and that $W|u(\cdot, 0+)| < \infty$ on D_+ . Then there exist a unique signed measure μ , supported in C and with finite total variation, and a unique initially zero temperature w on D_+ , such that $u = W\mu + Wu(\cdot, 0+) + w$ on D_+ .*

PROOF. Let ν be the initial measure of u , and let \mathcal{F} be an abundant Vitali covering of \mathbf{R}^n . Given $A, V \in \mathcal{F}$ such that $V \subseteq D(0) \setminus C$ and $A \cap V \neq \emptyset$, choose a bounded open set E such that $\overline{E} \subseteq D$ and $A \cap V \subseteq E(0)$. If $F = E \setminus (C \times \{0\})$, then $u = W\nu_F + v$ on F_+ for some initially zero temperature v , and $u(\cdot, 0+)$ is finite whenever $x \in F(0)$ and the limit exists. Therefore, by the note following Theorem 1,

$$\lim_{t \rightarrow 0+} \int_{A \cap V} u(x, t) dx = \int_{A \cap V} u(x, 0+) dx.$$

Since $W|u(\cdot, 0+)| < \infty$ on D_+ , $u(\cdot, 0+)$ is locally integrable on $D(0)$, and the result follows from Theorem 2. \square

REMARK. If $C = \bigcup_{j=1}^k \{x_j\}$ in Theorem 2, then

$$u(x, t) = \sum_{j=1}^k \alpha_j W(x - x_j, t) + W\psi(x, t) + w(x, t),$$

where $\alpha_j = \mu(\{x_j\})$ for all j . We can show that

$$\alpha_j = \lim_{t \rightarrow 0+} \int_{B(x_j, r)} (u(x, t) - \psi(x)) dx$$

for any r such that $\overline{B}(x_j, r) \subseteq D(0)$ and $\overline{B}(x_j, r) \cap C = \{x_j\}$. Given such an r , we have $|\mu|(\partial B(x_j, r)) = 0$, so that [13, Theorem 7.2(i)] implies that

$$\begin{aligned} \lim_{t \rightarrow 0+} \int_{B(x_j, r)} u(x, t) dx &= \lim_{t \rightarrow 0+} \int_{B(x_j, r)} (W\mu(x, t) + W\psi(x, t)) dx \\ &= \mu(B(x_j, r)) + \int_{B(x_j, r)} \psi(x) dx \\ &= \alpha_j + \int_{B(x_j, r)} \psi(x) dx, \end{aligned}$$

as asserted. Compare [2, Theorem 3].

3. The representation theorem

In this section, we prove an extension of Aronson's representation theorem [2, Theorem 3]. This requires a hypothesis of Dirichlet regularity, which is to be understood in the sense of [10], as that allows us a much wider class of open sets than does the usual potential-theoretic sense in [4]. We therefore recall the necessary definitions.

Let $(y, s) \in \partial D$. We call (y, s) an *abnormal boundary point*, and write $(y, s) \in ab(\partial D)$, if there is an open ball B centred at (y, s) such that $B \cap (\mathbf{R}^n \times]-\infty, s[) \subseteq D$. If B can be found such that $B \cap (\mathbf{R}^n \times]-\infty, s[) = B \cap D$, then (y, s) is of the *first kind*, and so belongs to $ab_1(\partial D)$; otherwise, it is of the *second kind*, and belongs to $ab_2(\partial D)$. The *essential boundary* $ess(\partial D)$ consists of all boundary points that are not in $ab_1(\partial D)$. If $(y, s) \in ess(\partial D)$, we put $D(y, s) = D \cap (\mathbf{R}^n \times]s, \infty[)$ if $(y, s) \in ab_2(\partial D)$, and $D(y, s) = D$ otherwise.

If D is bounded, then every continuous function $f : ess(\partial D) \rightarrow \mathbf{R}$ is resolvable. A point $(y, s) \in ess(\partial D)$ is called *regular* if

$$\lim_{\substack{(x,t) \rightarrow (y,s) \\ (x,t) \in D(y,s)}} S_f^D(x, t) = f(y, s)$$

for every continuous $f : ess(\partial D) \rightarrow \mathbf{R}$, where S_f^D denotes the generalized solution to the Dirichlet problem for f on D . The set D is called *regular* if every $(y, s) \in ess(\partial D)$ is regular.

We now give conditions which ensure that a temperature u on D_+ belongs to $H^\Delta(D_+)$. These will be used in the representation theorem. We write $D(t)$ for $\{x : (x, t) \in D\}$.

THEOREM 3. *Suppose that D_+ is bounded and Dirichlet regular, that u is a temperature on D_+ , and that there is a continuous function $\psi : ess(\partial D_+) \rightarrow \mathbf{R}$ such that*

$$(4) \quad \lim_{\substack{(x,t) \rightarrow (y,s) \\ (x,t) \in D_+(y,s)}} u(x, t) = \psi(y, s)$$

whenever $(y, s) \in ess(\partial D_+) \setminus D$. Then u can be extended to $\tilde{D}_+ = D_+ \cup ab(\partial D_+)$ by putting

$$(5) \quad u(y, s) = \lim_{(x,t) \rightarrow (y, s-)} u(x, t),$$

and if

$$(6) \quad \liminf_{t \rightarrow 0^+} \int_{\tilde{D}_+(t)} u^+(x, t) dx < \infty,$$

then $u \in H^\Delta(D_+)$ and the function

$$t \mapsto \int_{\tilde{D}_+(t)} u^+(x, t) dx$$

is bounded.

PROOF. Let $a = \sup\{t : D_+(t) \neq \emptyset\}$. Whenever $0 < c < a$, u is bounded on the set $E_c = D_+ \cap (\mathbf{R}^n \times]c, a[)$. For suppose that $\{x_j\}$ is a sequence in $D(c)$ such that $(x_j, c) \rightarrow (y_0, c) \in \partial D_+$. Then $(y_0, c) \in \text{ess}(\partial D_+)$, so that $u(x_j, c) \rightarrow \psi(y_0, c)$ by (4). Therefore $u(\cdot, c)$ is bounded, so that u is bounded on E_c , by (4) and the maximum principle [10, Theorem 2]. For any $(y, s) \in ab(\partial D_+)$, the boundedness of u on $E_{\frac{1}{2}s}$ implies that the limit in (5) exists and is finite [4, p. 274].

By [10, Theorem 32], ψ is resolutive for D_+ . Let $h = S_\psi^{D_+}$, and let $g = u - h$. Then (4) and the regularity of D_+ imply that

$$(7) \quad \lim_{\substack{(x,t) \rightarrow (y,s) \\ (x,t) \in D_+(y,s)}} g(x, t) = 0$$

whenever $(y, s) \in \text{ess}(\partial D_+) \setminus D$. Since h is bounded, g is bounded on E_c whenever $0 < c < a$, and therefore g can be extended to \tilde{D}_+ as u was. Since D_+ is bounded, it now follows from (6) that

$$\liminf_{t \rightarrow 0^+} \int_{\tilde{D}_+(t)} g^+(x, t) dx < \infty.$$

Put $w = g^+$ on \tilde{D}_+ , and $w = 0$ elsewhere on \mathbf{R}_+^{n+1} . Then w is continuous on $\mathbf{R}_+^{n+1} \setminus \partial D_+$, and also on $n(\partial D_+)$ because of (7). Furthermore, w is upper semicontinuous on $ab_1(\partial D_+)$, and also on $ab_2(\partial D_+)$ in view of (7). On $\mathbf{R}_+^{n+1} \setminus ab(\partial D_+)$, w satisfies locally the mean value inequality characteristic of subtemperatures. An application of Fatou's lemma shows that w also satisfies locally the mean value inequality at points of $ab(\partial D_+)$. Hence w is a subtemperature, by [8, Theorem 15].

Since g is bounded on each E_c , and D_+ is bounded, given $c \in]0, a[$ there is $\kappa_c < \infty$ such that

$$\int_{\tilde{D}_+(t)} g^+(x, t) dx \leq \kappa_c$$

whenever $c < t < a$. Therefore, if $0 < c < t < d < b$, then

$$\int_{\mathbf{R}^n} W(x, b-t)w(x, t) dx \leq (4\pi(b-d))^{-\frac{1}{2}n} \kappa_c.$$

Thus the function $t \mapsto \int_{\mathbf{R}^n} W(x, b-t)w(x, t)dx$ is locally bounded on $]0, b[$, for any $b > 0$. Furthermore,

$$\liminf_{t \rightarrow 0^+} \int_{\mathbf{R}^n} W(x, b-t)w(x, t)dx \leq (2\pi b)^{-\frac{1}{2}n} \liminf_{t \rightarrow 0^+} \int_{\tilde{D}_+(t)} g^+(x, t)dx < \infty.$$

Hence, in the notation of [9, Theorem 19], $w \in \Phi_b$ whenever $0 < b < \infty$, so that there is a temperature v which majorizes w on \mathbf{R}_+^{n+1} . Now $u - h \leq g^+ \leq v$ on D_+ , so that

$$u - h = v - (v - u + h) \in H^\Delta(D_+).$$

Since h is the generalized Dirichlet solution for ψ , we have $h \in H^\Delta(D_+)$, so that $u \in H^\Delta(D_+)$ as asserted.

For the last part, choose r such that $D_+(t) \subseteq B(0, r)$ for all $t \in]0, a[$, and choose $b \in]a, \infty[$. Then, whenever $0 < t < a$,

$$\begin{aligned} \int_{\tilde{D}_+(t)} u^+(x, t)dx &\leq \int_{\tilde{D}_+(t)} (g^+(x, t) + h^+(x, t))dx \\ &\leq \int_{\tilde{D}_+(t)} g^+(x, t)dx + \sup |h|v_n r^n, \end{aligned}$$

where v_n is the volume of the unit ball in \mathbf{R}^n . Furthermore,

$$\int_{\tilde{D}_+(t)} g^+(x, t)dx \leq (4\pi b)^{\frac{1}{2}n} \exp\left(\frac{r^2}{4(b-a)}\right) \int_{\mathbf{R}^n} W(x, b-t)w(x, t)dx,$$

and the integral on the right is bounded as a consequence of [9, Theorem 16]. \square

We can now prove our extension of Aronson's result. Here G_D denotes the Green function for D in the sense of [10], and

$$G_D \mu(x, t) = \int_D G_D(x, t; y, s) d\mu(y, s)$$

for a signed measure μ of finite total variation. If G is G_D with $D = \mathbf{R}^{n+1}$, and $\nu = \lambda \times \delta_0$ with δ_0 the unit mass at 0, then

$$G\nu(x, t) = \int_{\mathbf{R}^n \times \{0\}} W(x - y, t) d\nu(y, 0) = W\lambda(x, t)$$

whenever $t > 0$.

THEOREM 4. *Suppose that D_+ is bounded and Dirichlet regular, that C is a compact subset of $D(0)$, that \mathcal{F} is an abundant Vitali covering of \mathbf{R}^n , and that ψ is a continuous real-valued function on $\text{ess}(\partial D_+)$. If u is a temperature on D_+ such that*

$$(8) \quad \lim_{\substack{(x,t) \rightarrow (y,s) \\ (x,t) \in D_+(y,s)}} u(x,t) = \psi(y,s)$$

for every $(y,s) \in \text{ess}(\partial D_+) \setminus D$,

$$\liminf_{t \rightarrow 0^+} \int_{\tilde{D}_+(t)} u^+(x,t) dx < \infty,$$

and

$$\lim_{t \rightarrow 0^+} \int_{A \cap V} u(x,t) dx = \int_{A \cap V} \psi(x,0) dx$$

whenever $A, V \in \mathcal{F}$, $V \subseteq D(0) \setminus C$, and $A \cap V \neq \emptyset$, then

$$u = G_D v + S_\psi^{D_+}$$

on D_+ for some signed measure v of finite total variation supported in $C \times \{0\}$.

PROOF. By Theorem 3, $u \in H^\Delta(D_+)$. Therefore, by Theorem 2, there exist a signed measure μ , supported in C and with finite total variation, and an initially zero temperature w on D_+ , such that $u = W\mu + W\psi(\cdot, 0) + w$ on D_+ . Let $v = \mu \times \delta_0$, so that $Gv = W\mu$ on \mathbf{R}_+^{n+1} and $Gv = 0$ elsewhere. By the Riesz decomposition theorem, $Gv^+ = G_D v^+ + h_1$ and $Gv^- = G_D v^- + h_2$ on D , where h_1 and h_2 are the greatest thermic minorants of Gv^+ and Gv^- on D , so that each h_i is initially zero on D_+ . Thus, if $h = w + h_1 - h_2$ then $u - G_D v = W\psi(\cdot, 0) + h$ on D_+ . It follows from (8) and [11, Theorem 2] that (8) holds with u replaced by $u - G_D v$. Furthermore, if $v = W\psi(\cdot, 0) + h$ then $v(x,t) \rightarrow \psi(y,0)$ as $(x,t) \rightarrow (y,0+)$ whenever $(y,0) \in \text{ess}(\partial D_+) \cap D$. The result now follows from [10, Theorem 31]. \square

4. Removable singularities

We now consider the situation where u is a temperature on $D \setminus K$, for some compact set $K = C \times \{0\}$ with $C \subseteq D(0)$. In Theorem 5, under a mild constraint on u , we show that u is the sum of a temperature v on D and the potential Gv of a signed measure supported in K . Thus, if $v = \lambda \times \delta_0$ then $u = W\lambda + v$ on D_+ , and conditions which ensure that $W\lambda = 0$ become conditions for the removability of K . The proofs of Theorems 6, 7 and 8 all use this idea.

A temperature in $H^\Delta(D_+)$ is called *initially nonnegative* if its initial measure is nonnegative. Conditions that imply initial nonnegativity can be found in [15].

THEOREM 5. Let C be a compact subset of $D(0)$, let $K = C \times \{0\}$, and let u be a temperature on $D \setminus K$ such that

$$(9) \quad \liminf_{t \rightarrow 0^+} \int_U u^+(x, t) dx < \infty$$

for some open superset U of C in \mathbf{R}^n . Then u can be written uniquely as the sum of a temperature on D and the potential $G\nu$ of a signed measure supported in K .

PROOF. Let $\{V_k\}$ be an exhaustion of $U \cap D(0)$ by bounded open subsets of \mathbf{R}^n which are Dirichlet regular for Laplace's equation. Choose j such that $C \subseteq V_j$, and put $V = V_j$. Then \bar{V} is a compact subset of $D(0)$, so that we can find $a > 0$ such that the set $E = V \times] - a, a[$ has its closure in D . Note that E_+ is Dirichlet regular for the heat equation.

The essential boundary of $V \times] - a, 0[$ is a compact subset of $D \setminus K$, so that u is bounded there and hence also on $V \times] - a, 0[$. Therefore we can define a continuous, real-valued function ψ on $\text{ess}(\partial E_+)$ by putting

$$\psi(y, 0) = \lim_{(x,t) \rightarrow (y,0^-)} u(x, t)$$

for all $y \in V$, and

$$\psi(y, s) = u(y, s)$$

for all $(y, s) \in \partial V \times [0, a]$. Note that $\psi = u$ on $(V \setminus C) \times \{0\}$, and that $u(x, t) \rightarrow \psi(y, s)$ as $(x, t) \rightarrow (y, s)$ with $(x, t) \in E_+$, whenever $(y, s) \in \text{ess}(\partial E_+) \setminus E$. It follows from (9) and Theorem 3 that $u \in H^\Delta(E_+)$. Therefore, by the Corollary to Theorem 2, there exist a unique signed measure μ , supported in C and with finite total variation, and a unique initially zero temperature w on E_+ , such that $u = W\mu + W\psi(\cdot, 0) + w$ on E_+ . If

$$v = \begin{cases} W\psi(\cdot, 0) + w & \text{on } E_+, \\ u & \text{on } E \setminus E_+, \end{cases}$$

then v is continuous on E and a temperature on $E \setminus (V \times \{0\})$, so that v is a temperature on E , by [10, Theorem 5]. If $\nu = \mu \times \delta_0$, then $u = G\nu + v$ on E . Putting $v = u - G\nu$ on $D \setminus E$, we extend v to a temperature on D , and complete the proof. \square

THEOREM 6. Let C be a compact subset of $D(0)$ such that $m_n(C) = 0$, and let u be a temperature on $D \setminus (C \times \{0\})$ such that

$$\liminf_{t \rightarrow 0^+} \int_U u^+(x, t) dx < \infty$$

for some open superset U of C in \mathbf{R}^n . If there is an initially nonnegative $h \in H^\Delta(D_+)$ such that

$$(10) \quad \lim_{t \rightarrow 0^+} \frac{u(x, t)}{h(x, t)} = 0$$

for all $x \in C$ at which the limit exists, then u can be extended to a temperature on D .

PROOF. Since $m_n(C) = 0$, there exists a positive temperature f on \mathbf{R}_+^{n+1} such that $f(x, 0+) = \infty$ for all $x \in C$, by [11, Theorem 11]. The addition of f to h does not affect our hypotheses, and so we can assume that $h(x, 0+) = \infty$ for all $x \in C$.

By Theorem 5, there exist a signed measure μ supported in $C \times \{0\}$, and a temperature v on D , such that $u = G\mu + v$. Let $\mu = \lambda \times \delta_0$, and let v be the initial measure of h . By [14, Theorem 2],

$$(11) \quad \lim_{t \rightarrow 0^+} \frac{W\lambda(x, t)}{Wv(x, t)}$$

exists and is finite for v -almost all $x \in \mathbf{R}^n$. For each $x \in C$ at which this limit exists, the limit in (10) exists and the two are equal, because $h(x, 0+) = \infty$ for all $x \in C$. Therefore the limit in (11) is zero for v -almost all $x \in C$, and

$$\liminf_{t \rightarrow 0^+} \frac{|W\lambda(x, t)|}{Wv(x, t)} < \infty$$

for all $x \in C$. It now follows from [14, Theorem 6] (with $Z = C$ and $Y = \emptyset$) that λ_C is null. Hence μ is null, and $u = v$. \square

For the final two theorems, we denote by m_q the q -dimensional Hausdorff measure on \mathbf{R}^n , where $0 \leq q \leq n$. We are only concerned that a given set is null, finite, or σ -finite with respect to m_q , so there is no need to distinguish the case $q = n$ from Lebesgue measure.

THEOREM 7. *Let C be a compact subset of $D(0)$, and let u be a temperature on $D \setminus (C \times \{0\})$ such that*

$$\liminf_{t \rightarrow 0^+} \int_U u^+(x, t) dx < \infty$$

for some open superset U of C in \mathbf{R}^n . If either

(i) $q \in [0, n]$, $m_q(C) = 0$, and

$$(12) \quad \limsup_{t \rightarrow 0^+} t^{\frac{1}{2}(n-q)} |u(x, t)| < \infty \quad \text{for all } x \in C,$$

or

(ii) $q \in [0, n[$, C is σ -finite with respect to m_q , and

$$(13) \quad \lim_{t \rightarrow 0^+} t^{\frac{1}{2}(n-q)} u(x, t) = 0 \quad \text{for all } x \in C,$$

then u can be extended to a temperature on D .

PROOF. By Theorem 5, u can be written as the sum of a temperature v on D , and the potential $G\mu$ of a signed measure supported in $C \times \{0\}$. If $\mu = \lambda \times \delta_0$, then $u = W\lambda + v$ on D_+ . For any $x \in C$,

$$\limsup_{t \rightarrow 0^+} t^{\frac{1}{2}(n-q)} |v(x, t)|$$

is zero if $q < n$, and is finite if $q = n$. Therefore conditions (12) and (13) imply similar ones on $W\lambda$. We can now apply [14, Theorem 6] (with $Z = Y = C$, so that the auxiliary function is superfluous) and conclude that λ_C is null. Hence μ is null, and $u = v$. \square

In our final result we show that, if C has a certain structure, then for sets with finite m_q -measure we can weaken (13) without affecting the conclusion.

THEOREM 8. Let $q \in [0, n[$, let C be a compact subset of $D(0)$ such that $m_q(C) < \infty$ and

$$(14) \quad \liminf_{r \rightarrow 0} r^{-q} m_q(B(x, r) \cap C) > 0$$

for all $x \in C$, and let u be a temperature on $D \setminus (C \times \{0\})$ such that

$$\liminf_{t \rightarrow 0^+} \int_U u^+(x, t) dx < \infty$$

for some open superset U of C in \mathbf{R}^n . If

$$(15) \quad \liminf_{t \rightarrow 0^+} t^{\frac{1}{2}(n-q)} u(x, t) \leq 0 \leq \limsup_{t \rightarrow 0^+} t^{\frac{1}{2}(n-q)} u(x, t) \quad \text{for } m_q\text{-almost all } x \in C,$$

and

$$(16) \quad \liminf_{t \rightarrow 0^+} t^{\frac{1}{2}(n-q)} |u(x, t)| < \infty \quad \text{for all } x \in C,$$

then u can be extended to a temperature on D .

PROOF. By Theorem 5, $u = v + G(\lambda \times \delta_0)$ for some temperature v on E and signed measure λ supported in C . Since $q < n$, we have $t^{\frac{1}{2}(n-q)}v(x, t) \rightarrow 0$ as $t \rightarrow 0+$ for all $x \in C$, so that (15) and (16) imply similar conditions on $W\lambda$.

Suppose that $q > 0$. Then $m_q(C) < \infty$, (14) holds, $W\lambda(x, 0+) = 0$ for all $x \in \mathbf{R}^n \setminus C$ and m_n -a.e. on \mathbf{R}^n , and (15), (16) hold with $W\lambda$ in place of u , so that [12, Theorem 10] shows that $W\lambda = 0$. Hence $u = v$.

Now suppose that $q = 0$, so that C is finite. (In this case, (14) and (16) are superfluous.) Given $x_0 \in C$ such that $\lambda(\{x_0\}) > 0$, put $\nu = \lambda_{\{x_0\}}$ and $\omega = \lambda - \nu$. Then [14, Lemma 1] shows that $W\omega(x_0, t) = o(W\nu(x_0, t))$ as $t \rightarrow 0+$, so that

$$W\lambda(x_0, t) \sim W\nu(x_0, t) = (4\pi t)^{-\frac{1}{2}n}\lambda(\{x_0\}).$$

Since (15) holds with u replaced by $W\lambda$, it follows that

$$\lambda(\{x_0\}) = \lim_{t \rightarrow 0+} (4\pi t)^{\frac{1}{2}n} W\lambda(x_0, t) = 0,$$

a contradiction. Therefore $\lambda(\{x\}) = 0$ for all $x \in C$, so that again $u = v$. \square

REMARK. The conclusions of Theorems 7 (ii) and 8 both fail if $q = n$. For example, let $n = 1$, $D = [-1, 2]^2$, $C = [0, 1]$, $v(x, t) = e^{x+t} - (e-1)x - 1$ on D , and $u = v - Wv(\cdot, 0)_C$ on D_+ , $u = v$ on $D \setminus (D_+ \cup (C \times \{0\}))$. Then u is a bounded temperature, and because $v(\cdot, 0)$ is continuous on C with $v(0, 0) = v(1, 0) = 0$, we have $u(x, 0+) = 0$ for all $x \in C$. Since $u(x, 0-) = v(x, 0) < 0$ whenever $0 < x < 1$, u cannot be extended to a temperature on D .

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