J. Austral. Math. Soc. (*Series A*) **65** (1999), 326–332

DIRECTED GRAPHS AND NILPOTENT RINGS

A. V. KELAREV

(Received 3 September 1997; revised 10 September 1998)

Communicated by J. R. J. Groves

Abstract

Suppose that a ring is a sum of its nilpotent subrings. We use directed graphs to give new conditions sufficient for the whole ring to be nilpotent.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 16N40; secondary 16N60.

The investigation of rings which are sums of their subrings has been carried out by Bahturin and Giambruno [1], Bahturin and Kegel [2], Beidar and Mikhalev [3], Ferrero and Puczyłowski [5], Fukshansky [6], Herstein and Small [7], Kegel [8, 9], Kelarev and McConnell [13], Kepczyk and Puczyłowski [14, 15], Puczyłowski [16], Salwa [17] and the author [10–12]. Although there are several positive results which show that some properties are preserved by sums of two subrings, it turns out that relatively few ring-theoretic properties are inherited by rings which are sums of their two subrings, and there are no known nontrivial properties which are inherited by sums of three or more subrings.

A strong negative result of this sort was obtained by Bokut' [4]: Every algebra over a field of characteristic zero can be embedded in a simple algebra which is a sum of three nilpotent subalgebras. In [10] the author constructed a ring which is not nil but is a direct sum of two locally nilpotent subrings. A primitive ring which is a sum of two Wedderburn radical subrings was given in [11] with the use of a homomorphic image of the construction introduced in [10].

Therefore some additional restrictions on the interaction of the summands are needed in order to obtain positive results.

A natural restriction is to require that some products of the subrings are equal to zero. Suppose that a ring R is a sum of its subrings R_v, $v \in V$, and assume that for some pairs $u, v \in V$ it is known that the product $R_u R_v$ is equal to zero.

We use directed graphs (digraphs) to keep information about all pairs $u, v \in V$ with $R_u R_v = 0$. Denote by E the set of all ordered pairs $u, v \in V$ such that $R_u R_v = 0$, and consider the digraph $A = (V, E)$. By the *complement* of A we mean the digraph $G = \overline{A} = (V, \overline{E})$, where $\overline{E} = V \times V \backslash E$. Then we shall say that R is a *G-sum* of the R_v. The digraph A will be called the *annihilator digraph* of R.

This situation arises in several ring constructions. For example, if $U_n(F)$ is the ring of $n \times n$ upper triangular matrices over a field F, and e_{ij} denotes the standard matrix unit, then $U_n(F) = \oplus_{1 \le i \le j \le n} F e_{ij}$ and $F e_{ij} F e_{kl} = 0$ whenever $j \ne k$. For a finite set V, a ring R is a direct product of R_v, $v \in V$, if and only if R is a G-sum, where the annihilator digraph $A = \overline{G}$ is the complete digraph on V.

We say that a digraph $G = (V, E)$ is 2-connected if, for any $u, v \in V$, there exists $w \in V \backslash \{u, v\}$ such that $(u, w), (w, v) \in E$.

THEOREM 1. *Let G be a digraph without 2-connected subgraphs. If a ring R is a G-sum of nilpotent subrings, then R is nilpotent too.*

PROOF. Assume that the digraph $G(V, E)$ does not contain any 2-connected subgraphs. Take a ring R which is a G-sum of nilpotent subrings R_v, $v \in V$. Let R^1 be the ring R with identity 1 adjoined.

Put $H(R) = \cup_{v \in V} R_v$. For any $r \in H(R)$, we fix an element $\mathrm{ind}(r) \in V$ such that $r \in R_{\mathrm{ind}(r)}$.

If $U \subseteq V$, $|U| = k \ge 1$ and $m \ge k$, then by $L(U, m)$ we denote the set of all products of the form $s_1 t_1 s_2 t_2 \cdots s_k t_k s_{k+1}$ such that there exist positive integers $a_1, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k \ge m$, where $t_1, \ldots, t_k \in H(R)$, $t_l \in R^{a_l}_{\mathrm{ind}(t_l)}$ for $l = 1, \ldots, k$, $\{\mathrm{ind}(t_1), \ldots, \mathrm{ind}(t_k)\} = U$, and $s_1, \ldots, s_{k+1} \in R^1$.

For positive integers k, m, n, if $k > |V|$, then we put $P(k, m, n) = \{0\}$.

For positive integers k, m, n with $1 \le k \le |V|$ and $m \ge k$, denote by $P(k, m, n)$ the set consisting of zero and all products $r_1 r_2 \cdots r_n$ such that there exists a subset $U \subseteq V$ satisfying $|U| = k$ and $r_1, \ldots, r_n \in L(U, m)$.

We claim that every product in $P(k, m, 3(|V| + 1)n)$ is a sum of elements from $P(k, m + 1, n)$ and $P(k + 1, m + 1, n)$.

For $k > |V|$ the assertion is trivial. Assume that $k \le |V|$. Take any product $w = r_1 r_2 \cdots r_{3(|V|+1)n} \in P(k, m, 3(|V| + 1)n)$. By the definition of $P(k, m, 3(|V| + 1)n)$ there exists a subset $U \subseteq V$ such that $|U| = k$ and $r_1, \ldots, r_{3(|V|+1)n} \in L(U, m)$.

For any $i = 0, 1, \ldots, (|V| + 1)n - 1$, we rewrite the elements $r_{3i+1}, r_{3i+2}, r_{3i+3}$ and introduce an auxiliary set w_i which characterizes the way we rewrite them.

The definition of $L(U, m)$ shows that $r_{3i+j} = s_{j,1} t_{j,1} s_{j,2} t_{j,2} \cdots s_{j,k} t_{j,k} s_{j,k+1}$, for $j = 1, 2, 3$, where there exist positive integers $a_{j,1}, \ldots, a_{j,k}$ such that

$$a_{j,1} + a_{j,2} + \cdots + a_{j,k} \ge m,$$

$t_{j,1}, \dots, t_{j,k} \in H(R)$, $t_{j,l} \in R^{a_{j,l}}_{\mathrm{ind}(t_{j,l})}$ for $l = 1, \dots, k$, $\{\mathrm{ind}(t_{j,1}), \dots, \mathrm{ind}(t_{j,k})\} = U$, and $s_{j,1}, \dots, s_{j,k+1} \in R^1$.

If the three sets of pairs

$$\{(\mathrm{ind}(t_{j,1}), a_{j,1}), \dots, (\mathrm{ind}(t_{j,k}), a_{j,k})\}, \qquad j = 1, 2, 3$$

are not equal to each other, then for each value of $\mathrm{ind}(t_{j,l})$ we can choose the maximum power $a_{j,l}$ out of the three available powers $a_{j,l}$, $j = 1, 2, 3$. Since for some $\mathrm{ind}(t_{j,l})$ the elements $a_{j,l}$, $j = 1, 2, 3$ are not all equal, it follows that the sum of exponents of the chosen maximal powers is strictly greater than m. We keep the corresponding maximal elements $t_{j,l} \in R^{a_{j,l}}_{\mathrm{ind}(t_{j,l})}$ and multiply together the other elements which are between them. In this way we rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U, m+1)$. In this case we put $w_i = \emptyset$ to remember that $r_{3i+1}r_{3i+2}r_{3i+3}$ has been rewritten as a product in $L(U, m+1)$.

Next consider the case where all three sets of pairs

$$\{(\mathrm{ind}(t_{j,1}), a_{j,1}), \dots, (\mathrm{ind}(t_{j,k}), a_{j,k})\}, \qquad j = 1, 2, 3$$

are equal to each other. In this case we rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a sum of several elements, we consider only one summand and we introduce w_i to characterize this summand.

Given that the graph $G(V, E)$ is not 2-connected, we can find $u_1, u_2 \in U$ such that for any $w \in U \backslash \{u_1, u_2\}$ either $(u_1, w) \notin E$ or $(w, u_2) \notin E$. Then we can find l_1, l_2 such that $\mathrm{ind}(t_{2,l_1}) = u_1$ and $\mathrm{ind}(t_{3,l_2}) = u_2$. Let $r_{3i+2} = a_1 t_{2,l_1}^{a_{2,l_1}} b_1$ and $r_{3i+3} = a_2 t_{3,l_2}^{a_{3,l_2}} b_2$. Multiplying together $b_1 a_2$ we use the fact that $R = \bigoplus_{v \in V} R_v$ and represent the product as a sum $b_1 a_2 = \sum_{v \in V} c_v$, where $c_v \in R_v$. When we substitute the sum for $b_1 a_2$, the product $r_{3i+1}r_{3i+2}r_{3i+3}$ turns into a sum of several elements $r_{3i+1}a_1 t_{2,l_1} c_v t_{3,l_2} b_2$, where $v \in V$. We consider only one of these elements, for an arbitrary $v \in V$. Naturally, the product $r_1 \cdots r_{3(|V|+1)n}$ also becomes a sum of several summands, and we consider only one of these summands.

If $v = u_1$, then $t_{2,l_1} c_v \in R_{u_1}^{a_{2,l_1}+1}$. Using this we can rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U, m+1)$ and we put $w_i = \emptyset$.

If $v = u_2$, then $c_v t_{3,l_2} \in R_{u_2}^{a_{3,l_2}+1}$. Using this we can rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U, m+1)$ and we put $w_i = \emptyset$.

If $v \in U \backslash \{u_1, u_2\}$, then either $(u_1, v) \notin E$ or $(v, u_2) \notin E$. It follows that either $t_{2,l_1} c_v = 0$ or $c_v t_{3,l_2} = 0$, respectively. Therefore $r_{3i+1} a_1 t_{2,l_1} c_v t_{3,l_2} b_2 = 0$. In this case the corresponding summand of $r_1 \cdots r_{3(|V|+1)n}$ is zero and belongs to $P(k, m+1, n)$, as claimed.

If $v \in V \backslash U$, then we rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U \cup \{v\}, m+1)$ and we put $w_i = \{c_v\}$.

Thus all products $r_{3i+1}r_{3i+2}r_{3i+3}$ have been rewritten. Therefore the whole product $r_1 \cdots r_{3(|V|+1)n}$ has also been rewritten. We consider only one summand s of

$r_1 \cdots r_{3(|V|+1)n}$. The corresponding elements $w_1, \ldots, w_{(|V|+1)n}$ characterizing this summand s have been introduced.

Since the elements $w_1, \ldots, w_{(|V|+1)n}$ are chosen in $V \cup \{\emptyset\}$, there exist

$$1 \le i_1 < \cdots < i_n \le (|V|+1)n$$

such that $w_{i_1} = \cdots = w_{i_n} = w$.

If $w = \emptyset$, then all the summands of $r_{3i_l+1} r_{3i_l+2} r_{3i_l+3}$, $l = 1, \ldots, n$, which we considered, have been rewritten as elements of $L(U, m+1)$. Therefore we can rewrite the whole summand s as an element of $P(k, m+1, n)$, as claimed.

If $w = \{v\}$ for $v \in V$, then all the summands of $r_{3i_l+1} r_{3i_l+2} r_{3i_l+3}$, $l = 1, \ldots, n$, which we considered, have been rewritten as elements of $L(U \cup \{v\}, m+1)$. Therefore we can rewrite the whole summand s as an element of $P(k, m+1, n)$, as claimed.

Thus every product in $P(k, m, 3(|V|+1)n)$ is a sum of elements from $P(k, m+1, n)$ and $P(k+1, m+1, n)$.

Denote by N the maximum of the nilpotency indices of the rings R_v, $v \in V$. Then $R_v^N = 0$ for all v. Easy induction shows that every product in

$$P(1, 1, [3(|V|+1)]^{N|V|})$$

is a sum of elements from the sets $P(k, 1+N|V|, 1)$, for $1 \le k \le |V|$.

Take any element r in $P(k, 1+N|V|, 1)$. By the definition there exists a subset $U \subseteq V$ such that $|U| = k$ and $r \in L(U, 1+N|V|)$. Therefore $r = s_1 t_1 s_2 t_2 \cdots s_k t_k s_{k+1}$ and there exist positive integers $a_1, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k \ge 1 + N|V|$, where $t_1, \ldots, t_k \in H(R)$, $t_l \in R^{a_l}_{\mathrm{ind}(t_l)}$ for $l = 1, \ldots, k$, $\{\mathrm{ind}(t_1), \ldots, \mathrm{ind}(t_k)\} = U$, and $s_1, \ldots, s_{k+1} \in R^1$. We can choose a maximum exponent a_i for some $1 \le i \le k$. Clearly, $a_i \ge N$, and so $t_i \in R^{a_i}_{\mathrm{ind}(t_i)} = 0$. It follows that $r = 0$.

Thus $P(k, 1+N|V|, 1) = \{0\}$. Therefore $P(1, 1, [3(|V|+1)]^{N|V|}) = 0$.

Put $n = |V|\{[3(|V|+1)]^{N|V|} - 1\} + 1$, and consider an arbitrary product $w = r_1 \cdots r_n$, where $r_1, \ldots, r_n \in H(R)$. Since $\mathrm{ind}(r_i) \in V$ for all i, clearly there exist numbers

$$1 \le i_1 < i_2 < \cdots < i_{[3(|V|+1)]^{N|V|}} \le |V|\{[3(|V|+1)]^{N|V|} - 1\} + 1$$

such that

$$\mathrm{ind}(r_{i_1}) = \mathrm{ind}(r_{i_2}) = \cdots = \mathrm{ind}(r_{i_{[3(|V|+1)]^{N|V|}}}) = v.$$

Every element r_{i_j} belongs to $L(\{v\}, 1)$. Therefore w can be rewritten as a product in $P(1, 1, [3(|V|+1)]^{N|V|}) = 0$. Thus $H(R)^n = 0$, and so $R^n = 0$. □

COROLLARY 2. *For a graph $G = (V, E)$ the following conditions are equivalent:*

(i) *if a ring R is a G-sum of nilpotent subrings, then R is nilpotent too;*
(ii) *G does not contain triangles.*

PROOF. (i) $\Rightarrow$ (ii): Suppose that (ii) is not satisfied, that is G contains a triangle. Then Bokut's example of a ring which is not nilpotent but is a sum of three nilpotent subrings can be easily made a G-sum of the three nilpotent subrings and several zero subrings. Thus (i) does not hold. Thus (i) implies (ii).

(ii) $\Rightarrow$ (i): We can view the graph G as a digraph associating with every undirected edge two directed edges. Then it is easily seen that every 2-connected graph contains a triangle. Thus G does not contain 2-connected subgraphs by (ii). Theorem 1 yields (i). □

There exist directed graphs which are 2-connected but contain no triangles. For example, take $G = (V, E)$ with $V = \{O, A_1, \ldots, A_n\}$, where O is connected to all $A_1, \ldots, A_n$ by two-sided edges, each A_i is connected to A_{i+1} and A_n is connected to A_1 by directed edges.

Next, we discuss an example which shows that our Theorem 1 is probably not improvable. Let $G = (V, E)$ be a digraph containing a 2-connected digraph $H = (W, \overline{F})$ where $W \subseteq V$, $F \subseteq E$. We define a ring R which is an H-sum of subrings R_w, $w \in W$, with zero multiplication. If, after that, we put $R_v = 0$ for all $v \in V \backslash W$, then we see that R is a G-sum of the R_v. Hence we may throw out the vertices of G which do not belong to the 2-connected digraph H and assume that G is 2-connected from the very beginning. We also assume that E contains no loop (v, v), since we can throw away all loops from E without changing the 2-connectedness of G. Let $n = |V|$. To simplify further notation we assume that $V = \{1, \ldots, n\}$.

Let M be the set of terms formed by variables $x_1, \ldots, x_n$ with respect to n nonassociative operations $f_1, \ldots, f_n$. It can be defined recursively by the following two conditions:

(i) $x_1, \ldots, x_n \in M$;
(ii) $f_i(y, z)$ for all $y, z \in M$ and $i \in \{1, \ldots, n\}$.

For $i = 1, \ldots, n$, we define the sets

$$M_i = \{x_i\} \cup \{f_i(y, z) \mid y, z \in M\}.$$

Then $M = M_1 \cup \cdots \cup M_n$. For any $y \in M$, there exists an integer $\operatorname{ind}(y)$ such that $y \in M_{\operatorname{ind}(y)}$.

Let $\mathbb{R}$ be the field of real numbers. We define an $\mathbb{R}$-algebra R generated by the set M subject to relations

$$yz - f_1(y, z) - \cdots - f_n(y, z) = 0 \tag{1}$$

for all $y, z \in M$ such that $(\text{ind}(y), \text{ind}(z)) \in E$;

$$uv = f_1(u, v) = \cdots = f_n(u, v) = 0 \tag{2}$$

for all $u, v \in M$ such that $(\text{ind}(u), \text{ind}(v)) \notin E$.

For $i = 1, \ldots, n$, denote by R_i the subspace spanned over $\mathbb{R}$ by M_i. The relations (1) and (2) show that $R = \sum_{i=1}^{n} R_n$ is a G-sum. Given that E contains no loops (v, v), $v \in V$, it follows from (2) that all $R_1, \ldots, R_n$ are rings with zero multiplication.

Obviously, every 2-connected graph contains a directed cycle. Let $i_1, \ldots, i_k, i_1$ be a directed cycle in G. Then it seems that $w = (x_{i_1} \cdots x_{i_k})^m$ is nonzero for all positive integers m. The diamond lemma suggests itself as a tool for proving this.

In conclusion we look at the ring $SU_n(R)$ of strictly upper triangular matrices over any ring R to illustrate Theorem 1. Clearly, $SU_n(R) = \sum_{i<j} Re_{ij}$, where e_{ij} is the standard matrix unit. All the rings Re_{ij} have zero multiplication for $1 \leq i < j \leq n$. If we put $G = (V, E)$, where $V = \{(i, j) \mid 1 \leq i < j \leq n\}$ and $E = \{((i, j), (j, k)) \mid 1 \leq i < j < k \leq n\}$, then we see that $SU_n(R)$ is a G-sum of the rings Re_{ij}. It follows from Theorem 1 that $SU_n(R)$ is nilpotent.

References

[1] Yu. A. Bahturin and A. Giambruno, ‘Identities of sums of commutative subalgebras’ *Rend. Circ. Mat. Palermo (2)* **43** (1994)(2), 250–258.

[2] Yu. A. Bahturin and O. H. Kegel, ‘Lie algebras which are universal sums of abelian subalgebras’, *Comm. Algebra* **23** (1995), 2975–2990.

[3] K. I. Beidar and A. V. Mikhalev, ‘Generalized polynomial identities and rings which are sums of two subrings’, *Algebra i Logika* **34** (1995)(1), 3–11.

[4] L. A. Bokut’, ‘Embeddings in simple associative algebras’, *Algebra i Logika* **15** (1976)(2), 117–142.

[5] M. Ferrero and E. R. Puczyłowski, ‘On rings which are sums of two subrings’, *Arch. Math. (Basel)* **53** (1989), 4–10.

[6] A. Fukshansky, ‘The sum of two locally nilpotent rings may contain a non-commutative free subring’, *Proc. Amer. Math. Soc.*, to appear.

[7] I. N. Herstein and L. W. Small, ‘Nil rings satisfying certain chain conditions’, *Canad. J. Math.* **16** (1964), 771–776.

[8] O. H. Kegel, ‘Zur Nilpotenz gewisser assoziativer Ringe’, *Math. Ann.* **149** (1962/63), 258–260.

[9] O. H. Kegel, ‘On rings that are sums of two subrings’, *J. Algebra* **1** (1964), 103–109.

[10] A. V. Kelarev, ‘A sum of two locally nilpotent rings may be not nil’, *Arch. Math. (Basel)* **60** (1993), 431–435.

[11] A. V. Kelarev, ‘A primitive ring which is a sum of two Wedderburn radical subrings’, *Proc. Amer. Math. Soc.* **125** (1997), 2191–2193.

[12] A. V. Kelarev, ‘An answer to a question of Kegel on sums of rings’, *Canad. Math. Bull.* **41** (1998), 79–80.

[13] A. V. Kelarev and N. R. McConnell, ‘Two versions of graded rings’, *Publ. Math. (Debrecen)* **47** (1995) (3-4), 219–227.

[14] M. Kepczyk and E.R. Puczyłowski, 'On radicals of rings which are sums of two subrings', *Arch. Math. (Basel)* **66** (1996), 8–12.
[15] M. Kepczyk and E.R. Puczyłowski, 'Rings which are sums of two subrings', *J. Pure Appl. Algebra*, to appear.
[16] E. R. Puczyłowski, 'Some questions concerning radicals of associative rings', Theory of Radicals, Szekszárd, 1991, *Coll. Math. Soc. János Bolyai* **61** (1993), 209–227.
[17] A. Salwa, 'Rings that are sums of two locally nilpotent subrings', *Comm. Algebra* **24** (1996)(12), 3921–3931.

School of Mathematics
University of Tasmania
G.P.O. Box 252-37
Hobart, Tasmania 7001
Australia
e-mail: kelarev@hilbert.maths.utas.edu.au