DIRECTED GRAPHS AND NILPOTENT RINGS

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Abstract

Suppose that a ring is a sum of its nilpotent subrings. We use directed graphs to give new conditions sufficient for the whole ring to be nilpotent.


The investigation of rings which are sums of their subrings has been carried out by Bahturin and Giambruno [1], Bahturin and Kegel [2], Beidar and Mikhalev [3], Ferrero and Puczyłowski [5], Fukshansky [6], Herstein and Small [7], Kegel [8, 9], Kelarev and McConnell [13], Kepczyk and Puczyłowski [14, 15], Puczyłowski [16], Salwa [17] and the author [10–12]. Although there are several positive results which show that some properties are preserved by sums of two subrings, it turns out that relatively few ring-theoretic properties are inherited by rings which are sums of their two subrings, and there are no known nontrivial properties which are inherited by sums of three or more subrings.

A strong negative result of this sort was obtained by Bokut’ [4]: Every algebra over a field of characteristic zero can be embedded in a simple algebra which is a sum of three nilpotent subalgebras. In [10] the author constructed a ring which is not nil but is a direct sum of two locally nilpotent subrings. A primitive ring which is a sum of two Wedderburn radical subrings was given in [11] with the use of a homomorphic image of the construction introduced in [10].

Therefore some additional restrictions on the interaction of the summands are needed in order to obtain positive results.

A natural restriction is to require that some products of the subrings are equal to zero. Suppose that a ring $R$ is a sum of its subrings $R_v$, $v \in V$, and assume that for some pairs $u, v \in V$ it is known that the product $R_u R_v$ is equal to zero.
We use directed graphs (digraphs) to keep information about all pairs \( u, v \in V \) with \( R_u R_v = 0 \). Denote by \( E \) the set of all ordered pairs \( u, v \in V \) such that \( R_u R_v = 0 \), and consider the digraph \( A = (V, E) \). By the complement of \( A \) we mean the digraph \( G = \overline{A} = (V, \overline{E}) \), where \( \overline{E} = V \times V \setminus E \). Then we shall say that \( R \) is a \( G \)-sum of the \( R_v \). The digraph \( A \) will be called the annihilator digraph of \( R \).

This situation arises in several ring constructions. For example, if \( U_n(F) \) is the ring of \( n \times n \) upper triangular matrices over a field \( F \), and \( e_{ij} \) denotes the standard matrix unit, then \( U_n(F) = \oplus_{1 \leq i \leq j \leq n} F e_{ij} \) and \( F e_{ij} F e_{kl} = 0 \) whenever \( j \neq k \). For a finite set \( V \), a ring \( R \) is a direct product of \( R_v, v \in V \), if and only if \( R \) is a \( G \)-sum, where the annihilator digraph \( A = \overline{G} \) is the complete digraph on \( V \).

We say that a digraph \( G = (V, E) \) is 2-connected if, for any \( u, v \in V \), there exists \( w \in V \setminus \{u, v\} \) such that \((u, w), (w, v) \in E \).

**THEOREM 1.** Let \( G \) be a digraph without 2-connected subgraphs. If a ring \( R \) is a \( G \)-sum of nilpotent subrings, then \( R \) is nilpotent too.

**PROOF.** Assume that the digraph \( G(V, E) \) does not contain any 2-connected subgraphs. Take a ring \( R \) which is a \( G \)-sum of nilpotent subrings \( R_v, v \in V \). Let \( R^1 \) be the ring \( R \) with identity 1 adjoined.

Put \( H(R) = \bigcup_{v \in V} R_v \). For any \( r \in H(R) \), we fix an element \( \text{ind}(r) \in V \) such that \( r \in R_{\text{ind}(r)} \).

If \( U \subseteq V, |U| = k \geq 1 \) and \( m \geq k \), then by \( L(U, m) \) we denote the set of all products of the form \( s_1 t_1 s_2 t_2 \cdots s_k t_k s_{k+1} \) such that there exist positive integers \( a_1, \ldots, a_k \) satisfying \( a_1 + a_2 + \cdots + a_k \geq m \), where \( t_1, \ldots, t_k \in H(R), t_l \in R_{\text{ind}(t)} \) for \( l = 1, \ldots, k \), \( \{\text{ind}(t_1), \ldots, \text{ind}(t_k)\} = U \), and \( s_1, \ldots, s_{k+1} \in R^1 \).

For positive integers \( k, m, n \), if \( k > |V| \), then we put \( P(k, m, n) = \{0\} \).

For positive integers \( k, m, n \) with \( 1 \leq k \leq |V| \) and \( m \geq k \), denote by \( P(k, m, n) \) the set consisting of zero and all products \( r_1 r_2 \cdots r_n \) such that there exists a subset \( U \subseteq V \) satisfying \( |U| = k \) and \( r_1, \ldots, r_n \in L(U, m) \).

We claim that every product in \( P(k, m, 3(|V| + 1)n) \) is a sum of elements from \( P(k, m + 1, n) \) and \( P(k + 1, m + 1, n) \).

For \( k > |V| \) the assertion is trivial. Assume that \( k \leq |V| \). Take any product \( w = r_1 r_2 \cdots r_{3(|V| + 1)n} \in P(k, m, 3(|V| + 1)n) \). By the definition of \( P(k, m, 3(|V| + 1)n) \) there exists a subset \( U \subseteq V \) such that \( |U| = k \) and \( r_1, \ldots, r_{3(|V| + 1)n} \in L(U, m) \).

For any \( i = 0, 1, \ldots, (|V| + 1)n - 1 \), we rewrite the elements \( r_{3i+1}, r_{3i+2}, r_{3i+3} \) and introduce an auxiliary set \( w_i \) which characterizes the way we rewrite them.

The definition of \( L(U, m) \) shows that \( r_{3i+j} = s_j t_{j+1} s_j t_{j+2} \cdots s_j k t_{k+1} s_j k+1 \), for \( j = 1, 2, 3 \), where there exist positive integers \( a_{j,1}, \ldots, a_{j,k} \) such that

\[
a_{j,1} + a_{j,2} + \cdots + a_{j,k} \geq m,
\]
let $t_{j, 1}, \ldots, t_{j, k} \in H(R)$, $t_{j, l} \in R_{\text{ind}(t_{j, l})}^{a_{j, l}}$ for $l = 1, \ldots, k$, $\{\text{ind}(t_{j, 1}), \ldots, \text{ind}(t_{j, k})\} = U$, and $s_{j, 1}, \ldots, s_{j, k+1} \in R^1$.

If the three sets of pairs

$$\{(\text{ind}(t_{j, 1}), a_{j, 1}), \ldots, (\text{ind}(t_{j, k}), a_{j, k})\}, \quad j = 1, 2, 3$$

are not equal to each other, then for each value of $\text{ind}(t_{j, l})$ we can choose the maximum power $a_{j, l}$ out of the three available powers $a_{j, l}$, $j = 1, 2, 3$. Since for some $\text{ind}(t_{j, l})$ the elements $a_{j, l}$, $j = 1, 2, 3$ are not all equal, it follows that the sum of exponents of the chosen maximal powers is strictly greater than $m$. We keep the corresponding maximal elements $t_{j, l} \in R_{\text{ind}(t_{j, l})}^{a_{j, l}}$ and multiply together the other elements which are between them. In this way we rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U, m + 1)$. In this case we put $w_i = \emptyset$ to remember that $r_{3i+1}r_{3i+2}r_{3i+3}$ has been rewritten as a product in $L(U, m + 1)$.

Next consider the case where all three sets of pairs

$$\{(\text{ind}(t_{j, 1}), a_{j, 1}), \ldots, (\text{ind}(t_{j, k}), a_{j, k})\}, \quad j = 1, 2, 3$$

are equal to each other. In this case we rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a sum of several elements, we consider only one summand and we introduce $w_i$ to characterize this summand.

Given that the graph $G(V, E)$ is not 2-connected, we can find $u_1, u_2 \in U$ such that for any $w \in U \setminus \{u_1, u_2\}$ either $(u_1, w) \notin E$ or $(w, u_2) \notin E$. Then we can find $l_1, l_2$ such that $\text{ind}(t_{2, l_1}) = u_1$ and $\text{ind}(t_{3, l_2}) = u_2$. Let $r_{3i+2} = a_{1}t_{2, l_1}b_1$ and $r_{3i+3} = a_{2}t_{3, l_2}b_2$. Multiplying together $b_1a_2$ we use the fact that $R = \bigoplus_{v \in V} R_v$ and represent the product as a sum $b_1a_2 = \sum_{v \in V} c_v$, where $c_v \in R_v$. When we substitute the sum for $b_1a_2$, the product $r_{3i+1}r_{3i+2}r_{3i+3}$ turns into a sum of several elements $r_{3i+1}a_{1}t_{2, l_1}c_vt_{3, l_2}b_2$, where $v \in V$. We consider only one of these elements, for an arbitrary $v \in V$. Naturally, the product $r_{1} \cdots r_{3(|V|+1)n}$ also becomes a sum of several summands, and we consider only one of these summands.

If $v = u_1$, then $t_{2, l_1}c_v \in R_{a_{1}t_{2, l_1}+1}$. Using this we can rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U, m + 1)$ and we put $w_i = \emptyset$.

If $v = u_2$, then $c_vt_{3, l_2} \in R_{a_{2}t_{3, l_2}+1}$. Using this we can rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U, m + 1)$ and we put $w_i = \emptyset$.

If $v \in U \setminus \{u_1, u_2\}$, then either $(u_1, v) \notin E$ or $(v, u_2) \notin E$. It follows that either $t_{2, l_1}c_v = 0$ or $c_vt_{3, l_2} = 0$, respectively. Therefore $r_{3i+1}a_{1}t_{2, l_1}c_vt_{3, l_2}b_2 = 0$. In this case the corresponding summand of $r_{1} \cdots r_{3(|V|+1)n}$ is zero and belongs to $P(k, m + 1, n)$, as claimed.

If $v \in V \setminus U$, then we rewrite $r_{3i+1}r_{3i+2}r_{3i+3}$ as a product in $L(U \cup \{v\}, m + 1)$ and we put $w_i = \{c_v\}$.

Thus all products $r_{3i+1}r_{3i+2}r_{3i+3}$ have been rewritten. Therefore the whole product $r_{1} \cdots r_{3(|V|+1)n}$ has also been rewritten. We consider only one summand $s$ of
$r_1 \cdots r_{3(|V|+1)n}$. The corresponding elements $w_1, \ldots, w_{(|V|+1)n}$ characterizing this summand $s$ have been introduced.

Since the elements $w_1, \ldots, w_{(|V|+1)n}$ are chosen in $V \cup \{\emptyset\}$, there exist

$$1 \leq i_1 < \cdots < i_n \leq (|V| + 1)n$$

such that $w_{i_1} = \cdots = w_{i_n} = w$.

If $w = \emptyset$, then all the summands of $r_{3i_j+1}r_{3i_j+2}r_{3i_j+3}$, $l = 1, \ldots, n$, which we considered, have been rewritten as elements of $L(U, m+1)$. Therefore we can rewrite the whole summand $s$ as an element of $P(k, m+1, n)$, as claimed.

If $w = \{v\}$ for $v \in V$, then all the summands of $r_{3i_j+1}r_{3i_j+2}r_{3i_j+3}$, $l = 1, \ldots, n$, which we considered, have been rewritten as elements of $L(U \cup \{v\}, m+1)$. Therefore we can rewrite the whole summand $s$ as an element of $P(k, m+1, n)$, as claimed.

Thus every product in $P(k, m, 3(|V|+1)n)$ is a sum of elements from $P(k, m+1, n)$ and $P(k+1, m+1, n)$.

Denote by $N$ the maximum of the nilpotency indices of the rings $R_v$, $v \in V$. Then $R_v^N = 0$ for all $v$. Easy induction shows that every product in

$$P(1, 1, [3(|V| + 1)]^{N|V|})$$

is a sum of elements from the sets $P(k, 1 + N|V|, 1)$, for $1 \leq k \leq |V|$.

Take any element $r$ in $P(k, 1 + N|V|, 1)$. By the definition there exists a subset $U \subseteq V$ such that $|U| = k$ and $r \in L(U, 1 + N|V|)$. Therefore $r = s_1t_1s_2t_2 \cdots s_kt_k$ and there exist positive integers $a_1, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k \geq 1 + N|V|$, where $t_1, \ldots, t_k \in H(R)$, $t_l \in R^{a_l}_{\text{ind}(t_l)}$ for $l = 1, \ldots, k$, $\{\text{ind}(t_1), \ldots, \text{ind}(t_k)\} = U$, and $s_1, \ldots, s_k \in R^1$. We can choose a maximum exponent $a_i$ for some $1 \leq i \leq k$. Clearly, $a_i \geq N$, and so $t_i \in R^{a_i}_{\text{ind}(t_i)} = 0$. It follows that $r = 0$.

Thus $P(k, 1 + N|V|, 1) = \{0\}$. Therefore $P(1, 1, [3(|V| + 1)]^{N|V|}) = 0$.

Put $n = |V|[3(|V| + 1)]^{N|V|} - 1] + 1$, and consider an arbitrary product $w = r_1 \cdots r_n$, where $r_1, \ldots, r_n \in H(R)$. Since $\text{ind}(r_i) \in V$ for all $i$, clearly there exist numbers

$$1 \leq i_1 < i_2 < \cdots < i_{[3(|V|+1)]^{N|V|}} \leq |V|[3(|V| + 1)]^{N|V|} - 1] + 1$$

such that

$$\text{ind}(r_{i_1}) = \text{ind}(r_{i_2}) = \cdots = \text{ind}(r_{i_{[3(|V|+1)]^{N|V|}}}) = v.$$ 

Every element $r_{i_j}$ belongs to $L(\{v\}, 1)$. Therefore $w$ can be rewritten as a product in $P(1, 1, [3(|V| + 1)]^{N|V|}) = 0$. Thus $H(R)^n = 0$, and so $R^n = 0$. \hfill \Box

**Corollary 2.** For a graph $G = (V, E)$ the following conditions are equivalent:
(i) if a ring $R$ is a $G$-sum of nilpotent subrings, then $R$ is nilpotent too;
(ii) $G$ does not contain triangles.

PROOF. (i) $\Rightarrow$ (ii): Suppose that (ii) is not satisfied, that is $G$ contains a triangle. Then Bokut’s example of a ring which is not nilpotent but is a sum of three nilpotent subrings can be easily made a $G$-sum of the three nilpotent subrings and several zero subrings. Thus (i) does not hold. Thus (i) implies (ii).

(ii) $\Rightarrow$ (i): We can view the graph $G$ as a digraph associating with every undirected edge two directed edges. Then it is easily seen that every 2-connected graph contains a triangle. Thus $G$ does not contain 2-connected subgraphs by (ii). Theorem 1 yields (i).

There exist directed graphs which are 2-connected but contain no triangles. For example, take $G = (V, E)$ with $V = \{O, A_1, \ldots, A_n\}$, where $O$ is connected to all $A_1, \ldots, A_n$ by two-sided edges, each $A_i$ is connected to $A_{i+1}$ and $A_n$ is connected to $A_1$ by directed edges.

Next, we discuss an example which shows that our Theorem 1 is probably not improvable. Let $G = (V, E)$ be a digraph containing a 2-connected digraph $H = (W, F)$ where $W \subseteq V$, $F \subseteq E$. We define a ring $R$ which is an $H$-sum of subrings $R_w, w \in W$, with zero multiplication. If, after that, we put $R_v = 0$ for all $v \in V \setminus W$, then we see that $R$ is a $G$-sum of the $R_v$. Hence we may throw out the vertices of $G$ which do not belong to the 2-connected digraph $H$ and assume that $G$ is 2-connected from the very beginning. We also assume that $E$ contains no loop $(v, v)$, since we can throw away all loops from $E$ without changing the 2-connectedness of $G$. Let $n = |V|$. To simplify further notation we assume that $V = \{1, \ldots, n\}$.

Let $M$ be the set of terms formed by variables $x_1, \ldots, x_n$ with respect to $n$ nonassociative operations $f_1, \ldots, f_n$. It can be defined recursively by the following two conditions:

(i) $x_1, \ldots, x_n \in M$;
(ii) $f_i(y, z)$ for all $y, z \in M$ and $i \in \{1, \ldots, n\}$.

For $i = 1, \ldots, n$, we define the sets

$$M_i = \{x_i\} \cup \{f_i(y, z) \mid y, z \in M\}.$$ 

Then $M = M_1 \cup \cdots \cup M_n$. For any $y \in M$, there exists an integer $\text{ind}(y)$ such that $y \in M_{\text{ind}(y)}$.

Let $\mathbb{R}$ be the field of real numbers. We define an $\mathbb{R}$-algebra $R$ generated by the set $M$ subject to relations

$$yz - f_1(y, z) - \cdots - f_n(y, z) = 0$$

(1)
for all $y, z \in M$ such that $(\text{ind}(y), \text{ind}(z)) \in E$;

\begin{equation}
    uv = f_1(u, v) = \cdots = f_n(u, v) = 0
\end{equation}

for all $u, v \in M$ such that $(\text{ind}(u), \text{ind}(v)) \not\in E$.

For $i = 1, \ldots, n$, denote by $R_i$ the subspace spanned over $\mathbb{R}$ by $M_i$. The relations (1) and (2) show that $R = \sum_{i=1}^n R_n$ is a $G$-sum. Given that $E$ contains no loops $(v, v)$, $v \in V$, it follows from (2) that all $R_1, \ldots, R_n$ are rings with zero multiplication.

Obviously, every 2-connected graph contains a directed cycle. Let $i_1, \ldots, i_k, i_1$ be a directed cycle in $G$. Then it seems that $w = (x_{i_1} \cdots x_{i_k})^m$ is nonzero for all positive integers $m$. The diamond lemma suggests itself as a tool for proving this.

In conclusion we look at the ring $SU_n(R)$ of strictly upper triangular matrices over any ring $R$ to illustrate Theorem 1. Clearly, $SU_n(R) = \sum_{i<j} Re_{ij}$, where $e_{ij}$ is the standard matrix unit. All the rings $Re_{ij}$ have zero multiplication for $1 \leq i < j \leq n$. If we put $G = (V, E)$, where $V = \{(i, j) | 1 \leq i < j \leq n\}$ and $E = \{(i, j), (j, k) | 1 \leq i < j < k \leq n\}$, then we see that $SU_n(R)$ is a $G$-sum of the rings $Re_{ij}$. It follows from Theorem 1 that $SU_n(R)$ is nilpotent.

**References**


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