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A NOTE ON UNIFORM BOUNDS OF PRIMENESS IN MATRIX RINGS

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Abstract

A nonzero ring R is said to be uniformly strongly prime (of bound n) if n is the smallest positive integer such that for some n-element subset X of R we have $xXy \neq 0$ whenever $0 \neq x, y \in R$. The study of uniformly strongly prime rings reduces to that of orders in matrix rings over division rings, except in the case $n = 1$. This paper is devoted primarily to an investigation of uniform bounds of primeness in matrix rings over fields. It is shown that the existence of certain n-dimensional nonassociative algebras over a field F decides the uniform bound of the $n \times n$ matrix ring over F.

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1. Introduction and preliminary results

Unless stated otherwise, all rings are associative, but do not necessarily have an identity. A ring for which the associative law is not assumed to hold will always be referred to as a nonassociative ring. We denote by $\mathbb{M}_{n\times m}(R)$ the set of all $n \times m$ matrices over a ring R. If $n = m$ we write $\mathbb{M}_n(R)$ in place of $\mathbb{M}_{n\times m}(R)$. We use I_n to denote the identity matrix in $\mathbb{M}_n(R)$.

Let R be a ring and n, m positive integers. Let $X = \{A_1, A_2, \dots, A_m\}$ be an m-element subset of $\mathbb{M}_n(R)$. For $i = 1, 2, \dots, n$, let $\widehat{A_i}$ be the $n \times m$ matrix whose columns are the i-th columns of $A_1, A_2, \dots, A_m$. We obtain an n-element subset $\widehat{X} = \{\widehat{A}_1, \widehat{A}_2, \dots, \widehat{A}_n\}$ of $\mathbb{M}_{n\times m}(R)$. Note that if Y is an arbitrary n-element subset of $\mathbb{M}_{n\times m}(R)$ then it is possible to construct in the obvious fashion an m-element subset X of $\mathbb{M}_n(R)$ such that $Y = \widehat{X}$.

A ring R is said to be a *right order* in a ring S (with identity) if R is a subring of S such that (1) every element of R which is not a zero divisor of R is a unit of S, and (2) if $0 \neq s \in S$ then $0 \neq sr \in R$ for some $r \in R$.

Following Handelman and Lawrence [1, p. 211], we call a nonempty subset X of a ring R a *uniform insulator* for R if $xXy \neq 0$ whenever $0 \neq x, y \in R$. A nonzero ring R is said to be *uniformly strongly prime* if R contains a finite uniform insulator, and more specifically, *uniformly strongly prime of bound n*, if n is the smallest positive integer such that R possesses a uniform insulator of cardinality n.

This paper continues the investigation started in [5]. We address a single problem: determine the uniform bound of primeness of the matrix ring $\mathbb{M}_n(F)$ where F is a given field and n a positive integer. Our focus on matrix rings over fields is not as restrictive as it might appear, for every ring which is uniformly strongly prime of bound greater than 1 is isomorphic to a right or left order in a matrix ring over a division ring (see Theorem 1 below). Moreover, the uniform bound of primeness of an order in a matrix ring is, in many instances, equal to the bound of the over matrix ring (see Theorem 2 below). The task undertaken is also not as unambitious as it might appear. Indeed, we shall see that the uniform bound of primeness of the matrix ring $\mathbb{M}_n(F)$ is not determined solely by n, but also depends on subtle algebraic features of the ground field F.

THEOREM 1 ([5, Theorem 3]). *The following conditions are equivalent for a ring R:*

(i) *R is uniformly strongly prime of bound greater than* 1;
(ii) *R is isomorphic to a right or left order in $\mathbb{M}_n(D)$ for some division ring D and integer $n > 1$.*

It follows from the above theorem that rings which are uniformly strongly prime of bound greater than 1 are prime right or left Goldie. By contrast, rings which are uniformly strongly prime of bound precisely 1 need not be prime Goldie; a domain which is not Ore would be such an example.

THEOREM 2. (i) [4, Theorem 10] *If R is a right order in S and S is uniformly strongly prime of bound n then R is uniformly strongly prime of bound at most n.*
(ii) [5, Corollary 7] *If R is a right and left order in S then S is uniformly strongly prime of bound n if and only if R is uniformly strongly prime of bound n.*

Two sided orders do arise naturally as the following explanation shows. By the Faith-Utumi Theorem (see [2, p. 114]) a ring R is a right order in $\mathbb{M}_n(D)$ (D a division ring) if and only if there exists a ring embedding of R into $\mathbb{M}_n(D)$ and a right order C in D such that $\mathbb{M}_n(C)$ is contained in the image of R. It is an obvious consequence of the Faith-Utumi Theorem that if D is commutative then the right orders and left orders coincide in $\mathbb{M}_n(D)$. In this situation no ambiguity arises if we omit the prefixes 'right' and 'left' and speak simply of an order in $\mathbb{M}_n(D)$. It follows from Theorem 2 that *if F is a field and R is an order in $\mathbb{M}_n(F)$ then R and $\mathbb{M}_n(F)$ share the same uniform bound of primeness.*

The next result is an extension of [5, Lemma 4].

THEOREM 3. *Let D be a division ring and n, m positive integers. The following assertions are equivalent for an m-element subset $\{A_1, A_2, \dots, A_m\}$ of $\mathbb{M}_n(D)$:*

(i) *$\{A_1, A_2, \dots, A_m\}$ is a uniform insulator for $\mathbb{M}_n(D)$;*

(ii) *$x_1\widehat{A}_1 + x_2\widehat{A}_2 + \cdots + x_n\widehat{A}_n$ has trivial left annihilator in $\mathbb{M}_n(D)$ unless $x_1 = x_2 = \cdots = x_n = 0$;*

(iii) *$x_1\widehat{A}_1 + x_2\widehat{A}_2 + \cdots + x_n\widehat{A}_n$ has rank n unless $x_1 = x_2 = \cdots = x_n = 0$;*

(iv) *if W_D is the D-subspace of $\mathbb{M}_{n\times m}(D)$ spanned by $\widehat{A}_1, \widehat{A}_2, \dots, \widehat{A}_n$ then W_D has dimension n and every member of $W_D\backslash\{0\}$ has rank n.*

PROOF. (i) $\Leftrightarrow$ (ii) We use $\underline{x}$ to abbreviate $(x_1, x_2, \dots, x_n) \in D^n$ and a superscript T to denote the transpose of a matrix. Suppose (i) holds, then:

$$BA_i\underline{x}^T = \underline{0}^T \quad \text{for all } i \in \{1, 2, \dots, m\} \tag{1}$$

implies $B = 0$ whenever $B \in \mathbb{M}_n(D)$ and $\underline{x} \neq \underline{0}$. Equation (1) is equivalent to

$$B\left[A_1\underline{x}^T \,\middle|\, A_2\underline{x}^T \,\middle|\, \dots \,\middle|\, A_m\underline{x}^T\right] = 0 \quad (\text{in } \mathbb{M}_{n\times m}(D)). \tag{2}$$

But $\left[A_1\underline{x}^T \,\middle|\, A_2\underline{x}^T \,\middle|\, \dots \,\middle|\, A_m\underline{x}^T\right] = x_1\widehat{A}_1 + x_2\widehat{A}_2 + \cdots + x_n\widehat{A}_n$. Assertion (ii) follows. Reversing the above argument establishes (ii)$\Rightarrow$(i).

(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) is an immediate consequence of the fact that a matrix in $\mathbb{M}_{n\times m}(D)$ has trivial left annihilator in $\mathbb{M}_n(D)$ if and only if it has rank n. □

The next theorem follows from results in [5]. We include a direct proof for the sake of completeness.

THEOREM 4. *If D is a division ring then $\mathbb{M}_n(D)$ is uniformly strongly prime of bound m for some m satisfying $n \leq m \leq 2n - 1$.*

PROOF. If $m < n$ then no element of $\mathbb{M}_{n\times m}(D)$ has rank n. It follows from Theorem 3(iii) that $\mathbb{M}_n(D)$ cannot possess a uniform insulator of cardinality m.

It remains to show that $\mathbb{M}_n(D)$ has a uniform insulator of cardinality $2n - 1$. For each k satisfying $1 \leq k \leq 2n - 1$ define $A_k \in \mathbb{M}_n(D)$ to be the matrix whose entry in the i-th row and j-th column is 1 whenever $i + j = k + 1$, and is zero elsewhere.

Take $\underline{x} \in D^n$ and put

$$C = x_1\widehat{A}_1 + x_2\widehat{A}_2 + \cdots + x_n\widehat{A}_n = \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_n & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & x_1 & x_2 & \cdot & \cdot & \cdot & x_n & 0 & \cdot & \cdot & 0 \\ 0 & 0 & x_1 & x_2 & \cdot & \cdot & \cdot & x_n & 0 & \cdot & 0 \\ \cdot & & & \cdot & \cdot & & & & \cdot & & \\ \cdot & & & & \cdot & \cdot & & & & \cdot & \\ 0 & & & & & x_1 & x_2 & & & & x_n \end{bmatrix}.$$

Observe that $C \in \mathbb{M}_{n\times(2n-1)}(D)$. If $\operatorname{rank} C < n$ then the $n \times n$ submatrix of C consisting of columns 1 to n must be singular. This entails $x_1 = 0$. The $n \times n$ submatrix of C consisting of columns 2 to $n+1$ must also be singular which entails $x_2 = 0$; and so on. We thus obtain $x_1 = x_2 = \cdots = x_n = 0$. By Theorem 3, $\{A_1, A_2, \ldots, A_{2n-1}\}$ is a uniform insulator for $\mathbb{M}_n(D)$. □

Results of Section 2 show that the inequality $n \le m \le 2n-1$ of the previous theorem cannot be sharpened further.

We remind the reader that if D is a division ring and n a positive integer then $GL(n, D)$ denotes the set of all $n \times n$ matrices over D of rank n.

COROLLARY 5. *The following assertions are equivalent for a division ring D and positive integer n :*

(i) $\mathbb{M}_n(D)$ *is uniformly strongly prime of bound n;*
(ii) $GL(n, D) \cup \{0\}$ *contains an n-dimensional D-subspace of* $\mathbb{M}_n(D)$.

PROOF. (i) $\Rightarrow$ (ii) follows from Theorem 3 taking $m = n$.

(ii)$\Rightarrow$(i) Let W_D be a D-subspace of $\mathbb{M}_n(D)$ contained in $GL(n, D) \cup \{0\}$. Choose a basis $M_1, M_2, \ldots, M_n$ for W_D. Construct matrices $A_1, A_2, \ldots, A_n$ in $\mathbb{M}_n(D)$ such that $\widehat{A}_i = M_i$ for all $i \in \{1, 2, \ldots, n\}$. By Theorem 3, $\{A_1, A_2, \ldots, A_n\}$ is a uniform insulator for $\mathbb{M}_n(D)$.

Assertion (i) follows since, by Theorem 4, $\mathbb{M}_n(D)$ cannot possess a uniform insulator of cardinality less than n. □

If X is a uniform insulator for a ring R and e an idempotent in R then eXe is a uniform insulator for the ring eRe. If n, k are positive integers with $n \le k$ then it is possible to choose an idempotent $e \in \mathbb{M}_k(R)$ such that $\mathbb{M}_n(R) \cong e\mathbb{M}_k(R)e$. The next result follows.

PROPOSITION 6. *Let D be a division ring and n, k positive integers with $n \le k$. If $\mathbb{M}_k(D)$ is uniformly strongly prime of bound m then $\mathbb{M}_n(D)$ is uniformly strongly prime of bound at most m.*

2. Matrix rings over fields

The following theorem shows that the upper bound $2n-1$ of Theorem 4, cannot be lowered.

THEOREM 7 ([5, Proposition 8]). *If F is an algebraically closed field then $\mathbb{M}_n(F)$ is uniformly strongly prime of bound $2n-1$.*

The main results of this section show that the existence of certain n-dimensional nonassociative algebras over a field F decides the uniform bound of primeness of the $n \times n$ matrix ring over F. We need to recall some basic facts on nonassociative algebras.

Let F be a field. We call A_F a *nonassociative F-algebra* if A is an F-vector space endowed with a bilinear mapping from $A \times A$ to A with the property that $a(xy) = (ax)y = x(ay)$ whenever $a \in F$ and $x, y \in A$. We emphasize the fact that use of the term 'nonassociative' does not carry the assumption that the associative law fails to hold, but only that it is not assumed to hold. We say A_F has dimension n if A_F is an n-dimensional F-vector space.

Let A_F be a nonassociative F-algebra. For each $x \in A$ let ϕ_x denote the F-linear map on A defined by $\phi_x(y) = xy$ $(y \in A)$. The association $x \mapsto \phi_x$ defines an F-linear map from A_F into the (associative) F-algebra $\mathrm{End}_F\, A$. This mapping has kernel $\{x \in A : xA = 0\}$. We call the F-subalgebra of $\mathrm{End}_F\, A$ generated by $\{\phi_x : x \in A\}$ the *(left) enveloping algebra* of A_F, denoted $\mathscr{E}(A_F)$. Clearly if A_F has finite dimension n then $\mathscr{E}(A_F)$ may be interpreted as an F-subalgebra of $\mathbb{M}_n(F)$.

We call $a \in A_F$ *completely left* [*respectively right*] *invertible* if the equation $ax = b$ [respectively $xa = b$] has a unique solution for x whenever $b \in A$. If A is associative and possesses an identity element then the aforementioned notions coincide with that of a unit. We call A_F a *division algebra* if every nonzero $a \in A$ is completely left and right invertible. We point out that if A_F is finite dimensional then for A to be a division algebra it is sufficient that every nonzero $a \in A$ be completely left invertible. We shall denote by $U(A_F)$ the set of all completely left invertible elements of A_F.

THEOREM 8. *The following assertions are equivalent for a field F and positive integers n, m with $n \le m$:*

(i) *$GL(m, F) \cup \{0\}$ contains an n-dimensional F-subspace of $\mathbb{M}_m(F)$;*

(ii) *there exists a nonassociative F-algebra A_F for which $U(A_F) \cup \{0\}$ contains an n-dimensional F-subspace of A_F, and whose enveloping algebra $\mathscr{E}(A_F)$ is embeddable in $\mathbb{M}_m(F)$.*

PROOF. (i) $\Rightarrow$ (ii) Define $A_F = \mathbb{M}_m(F)$. Inasmuch as $\mathbb{M}_m(F)$ is an associative algebra with an identity element, $U(A_F) = GL(m, F)$ and A_F is isomorphic (as an F-algebra) to its enveloping algebra $\mathscr{E}(A_F)$. Therefore $\mathscr{E}(A_F)$ is embeddable in $\mathbb{M}_m(F)$.

(ii) $\Rightarrow$ (i) Let W_F be an n-dimensional F-subspace of A_F contained in $U(A_F) \cup \{0\}$. Let $\overline{W}_F$ denote the image of W_F in $\mathscr{E}(A_F)$. Since the canonical mapping from A_F to $\mathscr{E}(A_F)$ is F-linear, $\overline{W}_F$ is an F-subspace of $\mathscr{E}(A_F)$. Moreover, the kernel of this mapping (which is $\{x \in A : xA = 0\}$) has trivial intersection with W_F, so $\dim \overline{W}_F = n$. It follows from the definition of $U(A_F)$ that the image of $U(A_F)$ in $\mathscr{E}(A_F)$ is contained in the set of units of $\mathscr{E}(A_F)$. By hypothesis, $\mathscr{E}(A_F)$ is embeddable as an F-algebra in $\mathbb{M}_m(F)$. The image of $\overline{W}_F$ in $\mathbb{M}_m(F)$ is thus an n-dimensional F-subspace of $\mathbb{M}_m(F)$ contained in $GL(m, F) \cup \{0\}$. □

PROPOSITION 9. *Let F be a field and n, m positive integers with $n \leq m$. If $GL(m, F) \cup \{0\}$ contains an n-dimensional F-subspace of $\mathbb{M}_m(F)$, then $\mathbb{M}_n(F)$ is uniformly strongly prime of bound at most m.*

PROOF. Let W_F be an n-dimensional F-subspace of $\mathbb{M}_m(F)$ contained in $GL(m, F) \cup \{0\}$. Consider the F-linear projection mapping of $\mathbb{M}_m(F)$ onto $\mathbb{M}_{n\times m}(F)$ defined by

$$M \mapsto ZM \quad \text{where } Z = \left[I_n \,\middle|\, 0 \right] \in \mathbb{M}_{n\times m}(F).$$

Observe that $\operatorname{rank} ZM = \operatorname{rank} Z = n$ whenever $M \in GL(m, F)$. It follows that under this projection mapping the image of every member of $W_F \backslash \{0\}$ is a matrix with rank n. The result follows from Theorem 3. □

Observe that the special case $m = n$ of Proposition 9 is a consequence of Corollary 5.

REMARK 1. If m and n are distinct then the statement that $GL(m, F) \cup \{0\}$ contains an n-dimensional F-subspace of $\mathbb{M}_m(F)$ is, in general, strictly stronger than the statement that $\mathbb{M}_n(F)$ possesses a uniform insulator of cardinality m. Indeed, if F is an algebraically closed field, $A, B \in GL(m, F)$ and x an eigenvalue of AB^{-1}, then $A - xB \notin GL(m, F)$. This shows that for every positive integer m, $GL(m, F) \cup \{0\}$ cannot contain a 2-dimensional F-subspace of $\mathbb{M}_m(F)$. Yet, $\mathbb{M}_n(F)$ is uniformly strongly prime of bound $2n - 1$ for every algebraically closed field F and positive integer n (Theorem 7).

The following theorem follows from Corollary 5 and taking $m = n$ in Theorem 8.

THEOREM 10. *The following assertions are equivalent for a field F and positive integer n :*

(i) $\mathbb{M}_n(F)$ *is uniformly strongly prime of bound n;*

(ii) $GL(n, F) \cup \{0\}$ *contains an n-dimensional F-subspace of* $\mathbb{M}_n(F)$;

(iii) *there exists a nonassociative F-algebra A_F for which $U(A_F) \cup \{0\}$ contains an n-dimensional F-subspace of A_F, and whose enveloping algebra $\mathscr{E}(A_F)$ is embeddable in $\mathbb{M}_n(F)$.*

THEOREM 11. *Let F be a field. If there exists a nonassociative division F-algebra of dimension n, then $\mathbb{M}_n(F)$ is uniformly strongly prime of bound n.*

PROOF. If A_F is an n-dimensional division F-algebra, then $U(A_F) \cup \{0\} = A_F$. Moreover, since A_F is n-dimensional, $\mathscr{E}(A_F)$ is a subalgebra of $\mathbb{M}_n(F)$. The result then follows from Theorem 10. □

REMARK 2. It would be interesting to know whether the converse to Theorem 11 is valid.

COROLLARY 12. *Let F be a field which has an n-dimensional field extension E. Then $\mathbb{M}_n(F)$ is uniformly strongly prime of bound n.*

REMARK 3. We now describe a method for actually constructing a uniform insulator for $\mathbb{M}_n(F)$ given the existence of an n-dimensional nonassociative division F-algebra.

Choose a basis $u_1, u_2, \ldots, u_n$ for A_F. For each $i \in \{1, 2, \ldots, n\}$ let ϕ_{u_i} denote the F-linear mapping on A_F corresponding with left multiplication by u_i. Let $M_i \in \mathbb{M}_n(F)$ be the matrix representation of ϕ_{u_i} relative to the (ordered) basis $u_1, u_2, \ldots, u_n$. Choose a set of matrices $X = \{A_1, A_2, \ldots, A_n\}$ in $\mathbb{M}_n(F)$ such that $\widehat{X} = \{M_1, M_2, \ldots, M_n\}$. Then X is a uniform insulator for $\mathbb{M}_n(F)$.

Consider now the special case where $E = A_F$ is a simple field extension of F. Suppose $E = F(\alpha)$ where α has minimal polynomial $f(x) = a_0 + a_1 x + \cdots + x^n$ over F. If we choose $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ as a basis for E and the matrices $M_1, M_2, \ldots, M_n$ are constructed as above, then it is easily shown that $M_i = C^{i-1}$ for all $i \in \{1, 2, \ldots, n\}$ where

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & 1 \\ -a_0 & -a_1 & \cdot & \cdot & \cdot & -a_{n-1} \end{bmatrix}$$

is the so-called *companion matrix* of the polynomial $f(x)$.

THEOREM 13. *If F is an arbitrary finite field and n a positive integer, then $\mathbb{M}_n(F)$ is uniformly strongly prime of bound n.*

PROOF. If F is a finite field it must be a Galois field of order p^l for some prime p and positive integer l. The Galois field of order p^{ln} is a field extension of dimension n over F. The result follows from Corollary 12. □

We illustrate Theorem 13 with the following simple example.

EXAMPLE 1. Consider the two element field $\mathbb{Z}_2$. By Theorem 13, $\mathbb{M}_2(\mathbb{Z}_2)$ is uniformly strongly prime of bound 2. We shall exhibit a uniform insulator for $\mathbb{M}_2(\mathbb{Z}_2)$ following the procedure described in Remark 3.

The four element Galois field is a simple field extension of $\mathbb{Z}_2$ with minimal polynomial $f(x) = 1 + x + x^2$ and corresponding companion matrix $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Define $M_1 = A_1 = I_2$ and $M_2 = A_2 = C$. Observe that if $X = \{A_1, A_2\}$ then $\widehat{X} = \{M_1, M_2\}$. Thus $X = \{I_2, C\}$ is a uniform insulator for $\mathbb{M}_2(\mathbb{Z}_2)$.

EXAMPLE 2. Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and rational numbers respectively. Consider the polynomial $f(x) = x^n - p$ in $\mathbb{Z}[x]$ where n is an arbitrary positive integer and p a prime integer. (In fact we only require that p has no n-th root in $\mathbb{Z}$.) By the Eisenstein Irreducibility Criterion, $f(x)$ is irreducible in $\mathbb{Q}[x]$. If $\langle f(x)\rangle$ denotes the ideal of $\mathbb{Q}[x]$ generated by $f(x)$, then $F = \mathbb{Q}[x]/\langle f(x)\rangle$ is a (simple) field extension of dimension n over $\mathbb{Q}$. Hence $\mathbb{M}_n(\mathbb{Q})$ is uniformly strongly prime of bound n by Corollary 12. We exhibit a uniform insulator for $\mathbb{M}_n(\mathbb{Q})$.

The companion matrix of $f(x)$ is

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & 1 \\ p & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}.$$

Put $M_i = C^{i-1}$ for $i \in \{1, 2, \ldots, n\}$. Choose $X = \{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{M}_n(\mathbb{Q})$ such that $\widehat{X} = \{M_1, M_2, \ldots, M_n\}$. The set X is a uniform insulator for $\mathbb{M}_n(\mathbb{Q})$. A routine calculation shows that

$$A_1 = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & p \\ \cdot & & & & \cdot & 0 \\ \cdot & & & \cdot & 0 & \\ 0 & 0 & p & 0 & & \\ 0 & p & 0 & & & \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & p \\ \cdot & & & & \cdot & 0 \\ 0 & 0 & 0 & p & 0 & \\ 0 & 0 & p & 0 & & \end{bmatrix}, \ldots, A_n = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & & & & 1 & 0 \\ \cdot & & & \cdot & 0 & \\ \cdot & & \cdot & 0 & & \\ \cdot & 1 & 0 & & & \\ 1 & 0 & & & & \end{bmatrix}.$$

This example shows that the lower bound n of Theorem 4 cannot be raised.

The above example lends itself to the following generalization. Let D be a commutative domain with field of quotients Q. Suppose Q admits a nonarchimedean valuation φ. The Generalized Eisenstein Irreducibility Criterion asserts that if $\mathscr{P}_\varphi = \{a \in Q : \varphi(a) < 1\}$ and $f(x) = a_0 + a_1 x + \cdots + x^n$ is such that $\{a_1, a_2, \ldots, a_{n-1}\} \subseteq \mathscr{P}_\varphi$ and a_0 is not the product of two elements in $\mathscr{P}_\varphi$, then $f(x)$ is irreducible in $Q[x]$. (See [7, Exercise 2, p. 250].) The existence of such a polynomial $f(x)$ clearly implies that Q has an n-dimensional field extension whence $\mathbb{M}_n(Q)$ is uniformly strongly prime of bound n.

If, for example, D contains a prime element p such that $\bigcap_{n=1}^{\infty} \langle p \rangle^n = 0$ (this is the case for any prime p in a commutative noetherian domain), then φ can be chosen to be the $\langle p \rangle$-adic (nonarchimedean) valuation on Q. Relative to such a valuation φ the polynomial $f(x) = x^n - p$ satisfies the Generalized Eisenstein Irreducibility Criterion. We have thus proved the next theorem which extends [5, Proposition 9].

THEOREM 14. *Let D be a commutative domain and Q its field of quotients. If D contains a prime element p such that $\bigcap_{n=1}^{\infty} \langle p \rangle^n = 0$, then $\mathbb{M}_n(Q)$ is uniformly strongly prime of bound n for every positive integer n.*

If F is an arbitrary field and $F(x)$ the field of rational functions in x over F, then choosing $D = F[x]$ and $p = x$ in the previous theorem we obtain $\mathbb{M}_n(F(x))$ is uniformly strongly prime of bound n for every positive integer n.

We investigate now uniform bounds of primeness in matrix rings over the reals $\mathbb{R}$. The problem we face here is a difficult one and the results obtained are sketchy.

The complex numbers $\mathbb{C}$, the real quaternions $\mathbb{H}$ and the (nonassociative) octonions $\mathbb{O}$ are division algebras with respective dimensions 2, 4 and 8 over $\mathbb{R}$. It follows from Theorem 11 that $\mathbb{M}_n(\mathbb{R})$ is uniformly strongly prime of bound n for $n \in \{2, 4, 8\}$. We provide some details on the construction of a uniform insulator for $\mathbb{M}_8(\mathbb{R})$. We again follow the procedure described in Remark 3. Recall that the octonions $\mathbb{O}$ constitute an 8-dimensional algebra over $\mathbb{R}$ with basis $u_1, u_2, \ldots, u_8$ and multiplication induced by the following table:

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_2	u_2	$-u_1$	u_4	$-u_3$	u_6	$-u_5$	$-u_8$	u_7
u_3	u_3	$-u_4$	$-u_1$	u_2	u_7	u_8	$-u_5$	$-u_6$
u_4	u_4	u_3	$-u_2$	$-u_1$	u_8	$-u_7$	u_6	$-u_5$
u_5	u_5	$-u_6$	$-u_7$	$-u_8$	$-u_1$	u_2	u_3	u_4
u_6	u_6	u_5	$-u_8$	u_7	$-u_2$	$-u_1$	$-u_4$	u_3
u_7	u_7	u_8	u_5	$-u_6$	$-u_3$	u_4	$-u_1$	$-u_2$
u_8	u_8	$-u_7$	u_6	u_5	$-u_4$	$-u_3$	u_2	$-u_1$

For each $i \in \{1, 2, \ldots, 8\}$ let M_i be the matrix representation of ϕ_{u_i} relative to the basis $u_1, u_2, \ldots, u_8$. Then $M_1 = I_8$,

$$M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

All that remains is to construct a set of matrices $X = \{A_1, A_2, \ldots, A_8\}$ such that $\widehat{X} = \{M_1, M_2, \ldots, M_8\}$.

REMARK 4. Theorem 11 is unfortunately of limited value in determining the uniform bounds of primeness of matrix rings over $\mathbb{R}$, for it was proved by Bott and Milnor [3, Corollary 1] using deep results in algebraic topology that if A is an n-dimensional nonassociative division algebra over $\mathbb{R}$ then $n = 1, 2, 4$ or 8. It would be interesting to know whether these same results from algebraic topology yield restrictions on the dimensions of $\mathbb{R}$-subspaces of $\mathbb{M}_m(\mathbb{R})$ which are contained in $GL(m, \mathbb{R}) \cup \{0\}$. In view of Corollary 5 and Proposition 9, this would clearly throw some light on the uniform bounds of primeness of matrix rings over $\mathbb{R}$.

The embedding of the 4-dimensional quaternion $\mathbb{R}$-algebra into $\mathbb{M}_4(\mathbb{R})$ yields a uniform insulator for $\mathbb{M}_4(\mathbb{R})$ of cardinality 4. Uniform insulators are not unique, however. Indeed, the following example shows that it is possible to construct a uniform insulator for $\mathbb{M}_4(\mathbb{R})$ of cardinality 4 which does not derive from the subalgebra of quaternions.

EXAMPLE 3. Consider the matrices

$$I_4, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Then $x_1 I_4 + x_2 A + x_3 B + x_4 C \in GL(4, \mathbb{R})$ whenever $(x_1, x_2, x_3, x_4) \neq \underline{0}$, because $\operatorname{Det}(x_1 I_4 + x_2 A + x_3 B + x_4 C) = x_1^4 + 2x_1^2 x_2^2 + x_2^4 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_3^4 + x_2^2 x_3 x_4 - x_1^2 x_4^2 - x_2^2 x_4^2 + x_3^2 x_4^2 + x_4^4 = 0$ if and only if $x_1 = x_2 = x_3 = x_4 = 0$. If $X \subseteq \mathbb{M}_4(\mathbb{R})$ is chosen such that $\widehat{X} = \{I_4, A, B, C\}$ then X is a uniform insulator for $\mathbb{M}_4(\mathbb{R})$. It can be shown that the matrices I_4, A, B, C do not generate a proper subalgebra of $\mathbb{M}_4(\mathbb{R})$. In other words, I_4, A, B, C generate an algebra of dimension 16 over $\mathbb{R}$.

The following theorem is an assembly of all that is known on uniform bounds of primeness in matrix rings over $\mathbb{R}$.

THEOREM 15. (i) [5, p. 1162] *If n is an odd integer then $\mathbb{M}_n(\mathbb{R})$ is uniformly strongly prime of bound m for some m satisfying $n < m \leq 2n - 1$.*

(ii) [5, Example 2] $\mathbb{M}_2(\mathbb{R})$ *is uniformly strongly prime of bound* 2.

(iii) [5, Example 2] $\mathbb{M}_3(\mathbb{R})$ *and* $\mathbb{M}_4(\mathbb{R})$ *are uniformly strongly prime of bound* 4.

(iv) $\mathbb{M}_5(\mathbb{R})$ *is uniformly strongly prime of bound m for some m satisfying* $6 \leq m \leq 8$.

(v) $\mathbb{M}_6(\mathbb{R})$ *is uniformly strongly prime of bound m for some m satisfying* $6 \leq m \leq 8$.

(vi) $\mathbb{M}_7(\mathbb{R})$ *and* $\mathbb{M}_8(\mathbb{R})$ *are uniformly strongly prime of bound* 8.

PROOF. (i) In view of Theorem 4 it suffices to show that if n is odd then $\mathbb{M}_n(\mathbb{R})$ does not have a uniform insulator of cardinality n. Suppose, on the contrary, that $\{A_1, A_2, \ldots, A_n\}$ is a uniform insulator for $\mathbb{M}_n(\mathbb{R})$. Put $B = x_1\widehat{A}_1 + x_2\widehat{A}_2 + \cdots + x_n\widehat{A}_n$. Observe that $\operatorname{Det} B$ is a degree n homogeneous form in the variables $x_1, x_2, \ldots, x_n$. Inasmuch as every polynomial of odd degree over $\mathbb{R}$ has a root, it is not difficult to show that $\operatorname{Det} B = 0$ for some $\underline{x} = (x_1, x_2, \ldots, x_n) \neq \underline{0}$ (a detailed justification of this is provided in [6, p. 154]). This contradicts Theorem 3(iii).

(ii) The complex numbers constitute a 2-dimensional division $\mathbb{R}$-algebra. The result follows from Theorem 11.

(iii) The quaternions constitute a 4-dimensional division $\mathbb{R}$-algebra. By Theorem 11, $\mathbb{M}_4(\mathbb{R})$ is uniformly strongly prime of bound 4. This fact together with (i) above and Proposition 6 imply that $\mathbb{M}_3(\mathbb{R})$ is uniformly strongly prime of bound 4.

(vi) The octonions constitute an 8-dimensional division $\mathbb{R}$-algebra. By Theorem 11, $\mathbb{M}_8(\mathbb{R})$ is uniformly strongly prime of bound 8. This fact together with (i) above and Proposition 6 imply that $\mathbb{M}_7(\mathbb{R})$ is uniformly strongly prime of bound 8.

(iv) and (v) follow from arguments similar to those used in the proof of (iii) and (vi). □

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