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# ON THE LEBESGUE FUNCTION OF WEIGHTED LAGRANGE INTERPOLATION. II

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**Abstract**

The aim of this paper is to continue our investigation of the Lebesgue function of weighted Lagrange interpolation by considering Erdős weights on $\mathbb{R}$ and weights on $[-1, 1]$. The main results give lower bounds for the Lebesgue function on large subsets of the relevant domains.

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## 1. Introduction, notations and preliminary results

**1.1.** In [15] it was proved that the weighted Lebesgue function is 'big' on a 'large' subset of $[-a_n, a_n]$ for arbitrary fixed interpolatory matrix $X$ considering a class of Freud-type weights on $\mathbb{R}$. The aim of the present work is to extend this result for Erdős weights on $\mathbb{R}$ and for weights defined on $[-1, 1]$.

### 1A. Erdős weights on $\mathbb{R}$

**1.2.** DEFINITION. We say that $w \in \mathscr{E}(\mathbb{R})$ ($w$ is an Erdős weight on $\mathbb{R}$) if and only if $w(x) = e^{-Q(x)}$ where $Q\colon \mathbb{R} \to \mathbb{R}$ is even and is differentiable on $\mathbb{R}$, $Q' > 0$ and $Q'' \geq 0$ in $(0, \infty)$ and the function

$$T(x) := 1 + x\frac{Q''(x)}{Q'(x)}, \qquad x \in (0, \infty), \tag{1.1}$$

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is increasing in $(0, \infty)$, with

$$\lim_{x\to\infty} T(x) = \infty; \qquad T(0+) := \lim_{x\to 0+} T(x) > 1. \tag{1.2}$$

Moreover we assume that for some $C_1, C_2, C_3 > 0$

$$C_1 \le T(x)\frac{Q(x)}{xQ'(x)} \le C_2 \qquad \text{if} \qquad x \ge C_3 \tag{1.3}$$

(see [5, p. 201]).

The prototype of $w \in \mathscr{E}(\mathscr{R})$ is the case when $Q(x) = Q_{k,\alpha}(x) = \exp_k(|x|^\alpha)$, $k \ge 1, \alpha > 1$ where $\exp_k := \exp(\exp(\dots))$ denotes the $k$th iterated exponential. The corresponding $w$ will be denoted by $w_{k,\alpha}$. One can see that in that case

$$T(x) = \alpha x^\alpha \left\{\prod_{j=1}^{k-1} \exp_j(x^\alpha)\right\}(1 + o(1)), \qquad x \to \infty$$

(see [9, (1.8)]).

REMARK. We use the differentiability of $Q$ on the *whole* (open) line when we apply a result of Lubinsky [7, Lemma and Theorem 1] (see the 'Proof of Lemma 3.2' and 'Statement 3.5' of the present paper). Otherwise, evenness and conditions on the interval $(0, \infty)$ would be enough.

**1.3.** If $X \subset \mathbb{R}$ is an interpolatory matrix, that is

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < \infty, \qquad n \in \mathbb{N}, \tag{1.4}$$

for $f \in C(w, R)$ where $w \in \mathscr{E}(\mathscr{R})$ and

$$C(w, R) := \left\{ f : f \text{ is continuous on } \mathbb{R} \text{ and } \lim_{|x|\to\infty} f(x)w(x) = 0 \right\},$$

one can investigate the *weighted Lagrange interpolation* defined by

$$L_n(f, w, X, x) = \sum_{k=1}^{n} f(x_{kn})w(x_{kn})t_{kn}(w, X, x), \qquad n \in \mathbb{N}, \tag{1.5}$$

where

$$t_k(x) = t_{kn}(w, X, x) = \frac{w(x)}{w(x_{kn})} l_{kn}(X, x), \qquad 1 \le k \le n, \tag{1.6}$$

$$l_k(x) = l_{kn}(X, x) = \frac{\omega_n(X, x)}{\omega_n'(X, x_{kn})(x - x_{kn})}, \qquad 1 \le k \le n, \tag{1.7}$$

and

$$\omega_n(x) = \omega_n(X, x) = c_n \prod_{k=1}^{n} (x - x_{kn}), \qquad n \in N. \tag{1.8}$$

The polynomials $l_k$ of degree exactly $n-1$ (that is $l_k \in \mathscr{P}_{n-1} \setminus \mathscr{P}_{n-2}$ ) are the fundamental functions of the (usual) Lagrange interpolation while functions $t_k$ are the fundamental functions of the weighted Lagrange interpolation.

The classical Lebesgue estimation now has the form

$$|L_n(f, w, X, x) - f(x)w(x)| \le \{\lambda_n(w, X, x) + 1\} E_{n-1}(f, w) \tag{1.9}$$

where the (weighted) Lebesgue function is

$$\lambda_n(w, X, x) := \sum_{k=1}^{n} |t_{kn}(w, X, x)|, \qquad x \in \mathbb{R},\ n \in N \tag{1.10}$$

and

$$E_{n-1}(f, w) := \inf_{p \in \mathscr{P}_{n-1}} \|(f - p)w\|, \qquad n \in \mathbb{N}. \tag{1.11}$$

Here $\|\cdot\|$ is the sup norm on $\mathbb{R}$. If $w \in \mathscr{E}(\mathbb{R})$ then it is well-known that $E_{n-1}(f, w) \to 0$ if $n \to \infty$ and $f \in C(w, R)$.

Relation (1.9) and its immediate consequence

$$\|L_n(f, w, X) - fw\| \le \{\Lambda_n(w, X) + 1\} E_{n-1}(f, w), \tag{1.12}$$

where

$$\Lambda_n(w, X) := \|\lambda_n(w, X, x)\| \tag{1.13}$$

show that the investigation of $\lambda_n(w, X, x)$ and $\Lambda_n(w, X)$ (weighted Lebesgue constant) are fundamental. (For further motivations, see [15, §1].)

**1.4.** To get estimations for $\Lambda_n(w, X)$, at least for certain $X$, we consider the $n$ different roots

$$-\infty < y_{nn}(w^2) < y_{n-1,n}(w^2) < \cdots < y_{2n}(w^2) < y_{1n}(w^2) < \infty \tag{1.14}$$

of the $n$th orthonormal polynomial $p_n(w^2, x) \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$ with respect to $w^2 \in \mathscr{E}(\mathbb{R})$ (that is $\int_R p_n(w^2) p_m(w^2) w^2 = \delta_{nm}$). One can prove that for $Y(w^2) = \{y_{kn}(w^2)\}$ (see [1, (1.18)])

$$\Lambda_n(w, Y(w^2)) \sim (nT_n)^{1/6}, \qquad w \in \mathscr{E}(\mathbb{R}), \tag{1.15}$$

where $T_n \to \infty$ as $n \to \infty$. (Here, and later, $A_n \sim B_n$ means that $0 < c_1 \le A_n/B_n \le c_2$ where $c_1$ and $c_2$ do not depend on $n$, but may depend on other, previously fixed parameters.)

To be more precise about $T_n$, we introduce the corresponding Mhaskar–Rahmanov–Saff (MRS) number $a_u(w)$, the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 \frac{a_u t \;\, Q'(a_u t)}{\sqrt{1-t^2}}dt, \qquad u > 0 \tag{1.16}$$

(see [5, (1.13)]).
As an important application we mention the relations

$$\begin{cases} \|r_n w\| = \max\limits_{|x|\le a_n(w)} |r_n(x)w(x)| \\ \|r_n w\| > |r_n(x)w(x)| & \text{for } |x| > a_n(w) \end{cases} \tag{1.17}$$

valid for $r_n \in \mathscr{P}_n$ and $w \in \mathscr{E}(\mathbb{R})$.

If $w = w_{k,\alpha}$ then

$$a_n = \left\{\log_{k-1}\left(\log n - \frac{1}{2}\sum_{j=2}^{k+1}\log_{(j)} n + O(1)\right)\right\}^{1/\alpha} \tag{1.18}$$

where $\log_{(j)} = \log(\log(\dots))$, is the $j$th iterated logarithm.

Using $a_n$, $T_n$ can be written as

$$T_n = T(a_n(w)). \tag{1.19}$$

Later on we use that $T_n = o(n^2)$ (see [9, p. 209, (VIII)]).

Again, if $w = w_{k,\alpha}$, then $T_n \sim \prod_{j=1}^{k} \log_{(j)} n$ (see [9, (1.13)–(1.16)]).

**1.5.** But we can do better as far as the order of $\Lambda_n$ is concerned. Let $y_0 = y_{0n} > 0$ denote a point such that

$$|p_n(w^2, y_0)w(y_0)| = \|p_n(w^2)w\|. \tag{1.20}$$

Then if

$$V(w^2) = \{\{y_{kn}(w^2), 1 \le k \le n\} \cup \{y_{0n}, -y_{0n}\}, n \in N\}$$

one can prove the following.

Let $w \in \mathscr{E}(\mathbb{R})$. Then

$$\Lambda_n(w, V(w^2)) \sim \log n \tag{1.21}$$

(see [1, (1.22)]; concerning the additional points $\{\pm y_{0n}\}$, see [12]).

## 1B. Exponential weights on $[-1, 1]$

**1.6.** Instead of $\mathbb{R}$, we can define our weight function $w$ on the interval $(-1, 1)$. There is a substantial resemblance concerning formulas, definitions and theorems. So sometimes, especially in proofs, we only refer to the corresponding relations defined on $\mathbb{R}$. Following the exhaustive memoir of Levin and Lubinsky [4], we define the class of functions $W$ as follows.

DEFINITION. Let $w(x) = e^{-Q(x)}$ where $Q\colon (-1, 1) \to \mathbb{R}$, is even and is twice continuously differentiable in $(-1, 1)$. Assume moreover, that $Q' \geq 0$, $Q'' \geq 0$ in $(0, 1)$ and $\lim_{x \to 1-0} Q(x) = \infty$. The function

$$T(x) \colon = 1 + x\frac{Q''(x)}{Q'(x)}, \qquad x \in [0, 1) \tag{1.22}$$

is increasing in $[0, 1)$, moreover

$$\begin{cases} \text{(i)} & T(0+) > 1, \\ \text{(ii)} & T(x) \sim Q'(x)/Q(x), \quad x \text{ close enough to } 1, \\ \text{(iii)} & T(x)/(1 - x^2) \geq A > 2, \quad x \text{ close enough to } 1. \end{cases} \tag{1.23}$$

Then we write $w \in W$ (see [4, p. 5 and (1.34)]).

REMARKS. (1) Let $w_{0,\alpha}(x) = \exp(-(1 - x^2)^{-\alpha})$, $\alpha > 0$ and $w_{k,\alpha}(x) = \exp(-\exp_k(1 - x^2)^{-\alpha})$, $\alpha > 0$, $k \geq 1$. These strongly vanishing weights at $\pm 1$ are from $W$ ([4, §1]).
(2) Consider the ultraspherical Jacobi weight $w^{(\alpha)}(x) = (1 - x^2)^\alpha$, $\alpha > -1$. Here $Q(x) = -\alpha \log(1 - x^2)$, that is $w^{(\alpha)} \notin W$ if $-1 < \alpha < 0$ (the conditions for $Q(x)$ are not satisfied). If $\alpha \geq 0$ then $w^{(\alpha)}$ satisfies all the conditions required for $W$ but (1.23) (ii), (iii) (by routine calculation, $T(x) = 2(1 - x^2)^{-1}$ while $Q'(x)/Q(x) = -2x\{(1 - x^2)\log(1 - x^2)\}^{-1}$, $x \in (-1, 1)$). That means, $w^{(\alpha)} \notin W$ even for non-negative values of $\alpha$. However, they are very similar (at least from our point of view) to weights in $W$, so we can deal with them (see subsections 1.9 – 1.10).

**1.7.** Now the interpolatory matrix $X = \{x_{kn}\}$, $1 \leq k \leq n$, $n \in \mathbb{N}$, is in the open (!) interval $I = (-1, 1)$; the meaning of $C(w, I)$, $L_n(f, w, X, x)$, $\lambda_n(w, X, x)$, $\Lambda_n(w, X)$, $E_{n-1}(f, w)$, $p_n(w^2, x)$ and $\{y_{kn}(w^2)\} \subset (-1, 1)$ are clear (see (1.4)–(1.14)). For example if $w \in W$, then

$$C(w, I) \colon = \left\{ f : f \text{ is continuous on } I \text{ and } \lim_{|x| \to 1} f(x)w(x) = 0 \right\}.$$

Again, if $w \in W$, $E_{n-1}(f, w) \to 0$ whenever $f \in C(w, I)$, that is the Lebesgue estimation (1.12) holds true (now $\|\cdot\| = \max_{-1\le x\le 1} |\cdot|$). As one can prove

$$\Lambda_n(w, Y(w^2)) \sim (nT_n)^{1/6}, \qquad w \in W \tag{1.24}$$

(see [2]) where $T_n = T(a_n)$ and $a_n = a_n(w)$, $w \in W$, is defined by (1.16). By [4, (1.16), (1.17)], $1 - a_n(w_{0\alpha}) \sim n^{-1/(\alpha+\frac{1}{2})}$ and $1 - a_n(w_{k,\alpha}) \sim (\log_k n)^{-1/\alpha}$ whence, by (1.23) (iii), $T_n \to \infty$. On the other hand, by (1.23) (i) and [4, (3.8)], $1 < T_n = o(n^2)$.

**1.8.** As in subsection 1.5, using some additional points 'close' to $a_n(w)$, for the corresponding matrix $V(w^2)$ we get (see [2])

$$\Lambda_n(w, V(w^2)) \sim \log n, \qquad w \in W. \tag{1.25}$$

**1.9.** In subsections 1.9–1.10 we deal with Jacobi weights and their generalizations. First we give the rather general definition (see [10]; the present paper uses only a special case of [10; Definition 1.1]).

In what follows, $L^p[a,b]$ denotes the set of functions $F$ such that

$$\begin{cases} \|F\|_{L^p[a,b]} := \left\{ \int_a^b |F(t)|^p dt \right\}^{1/p} & \text{if} \quad 0 < p < \infty, \\ \|F\|_\infty := \operatorname{ess\,sup}_{a\le t\le b} |F(t)| & \text{if} \quad p = \infty \end{cases}$$

is finite. If $p \ge 1$ it is a norm; for $0 < p < 1$ its $p$th power defines a metric in $L^p[a,b]$.

By a *modulus of continuity* we mean a nondecreasing, continuous semiadditive function $\omega(\delta)$ on $[0,\infty)$ with $\omega(0) = 0$. If, in addition,

$$\omega(\delta) + \omega(\eta) \le 2\omega(\delta/2 + \eta/2) \qquad \text{for any } \delta, \eta \ge 0,$$

then $\omega(\delta)$ is a *concave* modulus of continuity, in which case $\delta/\omega(\delta)$ is nondecreasing for $\delta \ge 0$. We define $\omega(f,\delta)_p = \sup_{|\lambda|\le\delta} \|f(\lambda + \cdot) - f(\cdot)\|_p$, the *modulus of continuity of* $f$ in $L^p$ (where $L^p$ stands for $L^p[0, 2\pi]$).

For a fixed $m \ge 0$ let

$$-1 = u_{m+1} < u_m < \cdots < u_1 < u_0 = 1$$

and with $l_r \in \mathbb{N}$ $(r = 0, 1, \ldots, m+1)$

$$w_r(\delta) := \prod_{s=1}^{l_r} \{\omega_{rs}(\delta)\}^{\alpha(r,s)},$$

where $\omega_{rs}(\delta)$ are concave moduli of continuity with $\alpha(r,s) > 0$ ($s = 1, 2, \dots, l_r$; $r = 0, 1, \dots, m+1$).

Further let $H(x)$ be a *positive continuous* function on $[-1, 1]$ such that for $h(\vartheta) := H(\cos\vartheta)$

$$\omega(h,\delta)_\infty \delta^{-1} \in L^1[0,1] \quad \text{or} \quad \omega(h,\delta)_2 = 0(\sqrt{\delta}), \quad \delta \to 0.$$

DEFINITION. The function

$$w(x) = H(x) w_0(\sqrt{1-x}) w_{m+1}(\sqrt{1+x}) \prod_{r=1}^{m} w_r(|x-u_r|), \quad -1 \le x \le 1, \tag{1.26}$$

is a generalized Jacobi weight ($w \in GJ$), with singularities $u_r$ ($0 \le r \le m+1$).

REMARK. Since $\omega_{rs}(\tau) \le \omega_{rs}(\delta)$ ($0 \le \tau \le \delta$),

$$\int_0^\delta w_r(\tau)d\tau \le \delta w_r(\delta); \tag{1.27}$$

in [10, Definition 1.10] where $\alpha(r,s)$ might be negative, this important inequality had to be assumed (see [10, (1.12)]). Actually by (1.27) and [10, (1.24)] we get

$$\int_0^\delta w_r(\tau)d\tau \sim \delta w_r(\delta), \qquad r = 0, 1, \dots, m+1. \tag{1.28}$$

**1.10.** If $S(w) = S := \{u_r : r = 1, 2, \dots, m\}$ denotes the set containing the *inner* singularities of $w \in GJ$, a natural condition for an interpolatory $X \subset (1,1)$ is that $X \cap S = \emptyset$.

As above, one can define matrices $V(w^2) \subset (-1,1) \setminus S$, $w \in GJ$, with

$$\Lambda_n(w, V(w^2)) \sim \log n \tag{1.29}$$

(see [8], [11], [16]).

## 2. New results

**2.1.** It is natural to seek to prove that the order of the estimations $\Lambda(w, V(w^2)) \sim \log n$ (see (1.21), (1.25) and (1.29)) is the best amongst the interpolatory matrices. We can get much more.

THEOREM 2.1. *Let $w \in \mathscr{E}(\mathbb{R})$ and $0 < \varepsilon < 1$ be fixed. Then for any fixed interpolatory matrix $X \subset \mathbb{R}$ there exist sets $H_n = H_n(w, \varepsilon, X)$ with $|H_n| \le \varepsilon a_n(w)$ such that*

$$\lambda_n(w, X, x) > \frac{1}{3840}\varepsilon \log n \quad \text{if} \quad x \in [-a_n(w), a_n(w)] \setminus H_n, \tag{2.1}$$

*whenever $n \ge n_1$.*

REMARK. Here (and later) $n_1$ depends on $\varepsilon$ and $w$ but not on $X$.

**2.2.** Similarly on $(-1, 1)$ (see (1.25) and (1.29)), we state (with $S = \emptyset$ when $w \in W$) the following theorem.

THEOREM 2.2. *Let $w \in W \cup GJ$ and $0 < \varepsilon < 1$ be fixed. Then for any $X \subset (-1, 1) \setminus S$ there exist sets $H_n = H_n(w, \varepsilon, X)$ with $|H_n| \le \varepsilon$ such that*

$$\lambda_n(w, X, x) > \eta(\varepsilon, w) \log n \quad \text{if } x \in (-1, 1) \setminus H_n \tag{2.2}$$

*whenever $n \ge n_1$. Especially, $\eta(\varepsilon, w) = \varepsilon/3840$ if $w \in W$ or $w = (1 - x^2)^\alpha$, $\alpha \ge 0$.*

## 3. Proofs

**3.1.** PROOF OF THEOREM 2.1 (subsections 3.1–3.10). First we state some properties of $p_n = p_n(w^2)$ and $p_n w$, $w \in \mathscr{E}(\mathscr{R})$.

Let $0 < \varepsilon < 1$ be fixed and consider the interval $I_n = I_n(\varepsilon) = [-b_n, b_n] = [-a_n(1 - \varepsilon/5), a_n(1 - \varepsilon/5)]$. By definition $|[-a_n, a_n] \setminus I_n| = 2\varepsilon a_n/5$. First we deal with the interval $I_n$.

By (1.14), $p_n(x) = p_n(w^2, x) = \gamma_n(w^2) \prod_{k=1}^{n} (x - y_{kn}(w^2))$. Using the notation $y_{kn} = y_{kn}(w^2)$, we have

STATEMENT 3.1. Let $w \in \mathscr{E}(\mathbb{R})$. Then uniformly in $k$ and $n \in \mathbb{N}$

$$\widetilde{c}_1 \frac{a_n}{n} \le y_{kn} - y_{k+1,n} \le c_1 \frac{a_n}{n}, \qquad y_{k,n}, y_{k+1,n} \in I_n, \tag{3.1}$$

$$|p_n'(y_{kn}) w(y_{kn})| \sim \frac{n}{a_n^{3/2}}, \qquad y_{kn} \in I_n. \tag{3.2}$$

Moreover, uniformly in $k$, $x$ and $n \in \mathbb{N}$

$$|p_n(x) w(x)| \le c|x - y_{kn}| \frac{n}{a_n^{3/2}}; \qquad x, y_{kn} \in I_n. \tag{3.3}$$

Finally,

$$|p_n(x) w(x)| \le c\, a_n^{-1/2} (nT_n)^{1/6}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{3.4}$$

See [5, (1.24) and the remark after the formula] for (3.1); [5, last formula on p. 285] for (3.2); [5, (10.28)] for (3.3), and [5, (1.26)] for (3.4). We used that $\psi_n(x) \sim \varphi_n(x) \sim 1$ whenever $x \in I_n$. ($\psi_n(x)$ and $\varphi_n(x)$ are defined by [5; (1.19) and (10.11), (10.12)], respectively.)

Now let $y_j = y_{jn} = y_{j(n,x),n}$ be defined by

$$|x - y_{jn}| = \min_{1 \le k \le n} |x - y_{kn}|. \tag{3.5}$$

LEMMA 3.2. *We have, uniformly in $x \in I_n$,*

$$|p_n(x)w(x)| \sim |p_n'(y_{jn})w(y_{jn})|\,|x - y_{jn}| \sim \frac{n}{a_n^{3/2}}|x - y_{jn}|. \tag{3.6}$$

REMARKS. (1) The constants in formula (3.1)–(3.3) and (3.6) do depend on $\varepsilon$.
(2) By definition, (3.5) and (3.6) mean that $|(t_{jn}(Y(w^2), x)| \sim 1$ whenever $x \in I_n$.

PROOF OF LEMMA 3.2. Using [1, (2.16)],

$$\|t_{kn}(Y(w^2))\| \le c, \qquad 1 \le k \le n, \quad n \in \mathbb{N}. \tag{3.7}$$

Consider the polynomial $\tau_{kn}(x) = l_{kn}(Y(w^2), x)w^{-1}(y_k) \in \mathscr{P}_{n-1}$. By definition, $t_{kn}(x) = \tau_k(y_k)w(y_k) = 1$; further, using (3.7) we get $|\tau_k(x)w(x)| \le c$ for any $k$, $n$ and $x \in \mathbb{R}$. Then, applying a Markov–Bernstein inequality in [6, (1.26)],

$$\begin{aligned} |t_k(x)| &= |\tau_k(x)w(x)| = |\tau_k(y_k)w(y_k) + (\tau_k(\xi)w(\xi))'(x - y_k)| \\ &\ge |1 - c\,\eta\, n\, a_n^{-1} \cdot a_n n^{-1}| \ge 1/2 \quad \text{if} \quad |x - y_k| \le \eta a_n/n \end{aligned} \tag{3.8}$$

($\xi$ between $x$ and $y_k$, $x, y_k \in I_n$), whenever we choose $\eta > 0$, fixed, properly small.

Notice that $\eta > 0$ does not depend on $k$ and $n$.

Now, relations (3.7) and (3.8) give (3.6) at least for $x$ satisfying relations $|x - y_j| \le \eta\, a_n/n$, $x \in I_n$.

We can finish the proof of the lemma as follows. For a fixed $l$, denote by $z$ the unique maximum point in $(y_l, y_{l-1})$ of $|p_n(x)w(x)|$, $2 \le l \le n$ (for uniqueness consult Lubinsky [7, Lemma]). Using (3.3) if $x \in (y_l, y_{l-1}) \subset I_n$ and $k = l$, gives that $|p_n(z)w(z)| \le c\, a_n n^{-1} n\, a_n^{-3/2} \sim a_n^{-1/2}$. On the other hand if $z_1 = y_l + \eta\, a_n/n$, $z_2 = y_{l-1} - \eta\, a_n/n$, we get relations $|p_n(z_i)w(z_i)| \sim a_n n^{-1} n\, a_n^{-3/2} = a_n^{-1/2}$ (see (3.6)), whence $y_{l-1} - z \sim z - y_l \sim a_n/n$ is obvious. Then, we can choose $\eta > 0$ so that $z - z_1 \sim z_2 - z \sim a_n/n$. Now, if $x \in (z_1, z_2)$, by the monotonicity of $p_n w$ (see [7, Lemma]), $a_n^{-1/2} \sim |p_n(z)w(z)| \ge |p_n(x)w(x)| > \min(|p_n(z_1)w(z_1)|,\ |p_n(z_2)w(z_2)|) \sim a_n^{-1/2}$ which, using that now $|x - y_j| \sim a_n/n$, gives relation (3.6).

**3.2.** Next, we prove Theorem 2.1 for $x \in I_n = I_n(\varepsilon)$. Fix $n$ and let $K_n = \{k : x_{kn} \in I_n\}$. First suppose that $|K_n| := N = N_n > 0$ and denote the corresponding nodes $\{x_{kn}\} \subset I_n$ by $z_{1n}, z_{2n}, \dots, z_{Nn}$. We order them as

$$z_{N+1,n} := -b_n \le z_{Nn} < z_{N-1,n} < \cdots < z_{2n} < z_{1n} \le z_{0n} := b_n. \tag{3.9}$$

We introduce some other notations and definitions. Let

$$\begin{cases} J_k = J_{kn}(Z) := [z_{k+1,n}, z_{kn}], \quad (J_k) := (J_{kn}(Z)) = (z_{k+1,n}, z_{kn}), \\ J_k(q_k) = J_{kn}(q(J_{kn})) := [z_{k+1} + q_k|J_k|, z_k - q_k|J_k|], \\ \overline{J_k} = \overline{J_k(q_k)} := J_k \setminus J_k(q_k) \text{ with } 0 < q_k \le \frac{1}{2} \text{ and } 0 \le k \le N. \end{cases} \tag{3.10}$$

The interval $J_k$ is called *short* if and only if $|J_k| \le a_n\delta_n$, where $\delta_n = n^{-1/6}$, say; the others are called *long*. (Actually, arbitrary $\delta_n = n^{-\alpha}$, $0 < \alpha < 1$, works.)

**3.3.** For the long intervals we prove (see [15, Lemma 3.3] and the references there).

LEMMA 3.3. *Let* $w \in \mathscr{E}(\mathbb{R})$, $J_k \subset I_n$, $a_n\delta_n < |J_k|$, $c_0/(n\delta_n) < q_k < \frac{1}{4}$ *and define* $\varrho = \varrho(k,n) := [(q_k/2)|J_k|(n/c_1a_n)]$. *Then for a proper* $h_{kn} \subset J_k$ *we have*

$$\lambda_n(w, X, x) > c_2 \frac{3^{\varrho(k,n)}}{n^{7/6}T_n^{1/6}\delta_n} \qquad \text{if } x \in J_{kn} \setminus h_{kn}. \tag{3.11}$$

*Here* $|h_{kn}| \le 4q_k|J_k|$, $0 \le k \le N$, $n \ge n_0$; *the constants* $n_0$ *and* $c_0$ *are properly chosen.*

PROOF. Let us consider those roots $y_{in}$ of $p_n(x)$ which are in $J_k(q_k)$. By (3.1), their number is not less than

$$\left[(1 - 2q_k)|J_k|\frac{n}{c_1a_n}\right] > c(1 - 2q_k)n\delta_n.$$

Let us define the set $h_k = h_{kn}$ by

$$h_k = \overline{J_k(q_k)} \cup \left\{ \bigcup_{\Delta_i \subset J_k(q_k)} \overline{\Delta_i(q_k)} \right\},$$

where $\Delta_i = \Delta_i(Y) = [y_i, y_{i+1}]$ and $(\Delta_i)$, $\Delta_i(q_k)$, $\overline{\Delta_i}$ are defined according to (3.10). (We use the same $q_k = q(J_k)$ for every $\Delta_i$.) By construction,

$$|h_k| < 4q_k|J_k|.$$

To prove (3.11), let $y \in J_k \setminus h_k = J_k(q_k) \setminus h_k$ and consider the interval

$$M(y) = \left[y - \frac{q_k}{4}|J_k|, y + \frac{q_k}{4}|J_k|\right] \subset J_k\left(\frac{3q_k}{4}\right),$$

containing at least

$$\left[\frac{q_k}{2}|J_k|\frac{n}{c_1 a_n}\right] = \varrho > c\ q_k n\delta_n \geq 1 \tag{3.12}$$

roots of $p_n(x)$ if $c_0 > 0$ is properly chosen.

Consider the polynomial $r(x) = \prod_{y_i \notin M(y)}(x - y_i)$. Since

$$p_n(u) = \gamma_n r(u) \prod_{y_i \in M(y)} (u - y_i),$$

we have

$$w(x)r(x) = \frac{w(x)p_n(x)}{w(y)p_n(y)} w(y)r(y) \prod_{y_i \in M(y)} \frac{y - y_i}{x - y_i}.$$

Here, if $x \notin (J_k)$, by construction

$$\left|\frac{y - y_i}{x - y_i}\right| \leq \frac{1}{3};$$

$$|w(x)p_n(x)| \leq c\ a_n^{-1/2}(nT_n)^{1/6}$$

(see (3.4)). Finally if $y_i = y_j(y)$ is the nearest root of $p_n$ to $y$, by construction,

$$|w(y)p_n(y)| \geq c|p_n'(y_j)w(y_j)(y - y_j)| \sim na_n^{-3/2}q_k\frac{a_n}{n} = q_k a_n^{-1/2}$$

(see (3.6)). So, as $c_0 q_k^{-1} < n\,\delta_n$, we get

$$\begin{aligned} |w(x)r(x)| &\leq c|w(y)r(y)|\frac{a_n^{-1/2}(nT_n)^{1/6}}{q_k a_n^{-1/2}}3^{-\varrho} \\ &\leq c|w(y)r(y)|\frac{n\,\delta_n(nT_n)^{1/6}}{3^{\varrho}}, \qquad x \notin (J_k). \end{aligned} \tag{3.13}$$

On the other hand, since $\varrho \geq 1$, $r(x) \in \mathscr{P}_{n-1}$ whence, using Lagrange interpolation,

$$w(y)r(y) = \sum_{i=1}^{n} w(x_i)r(x_i)\frac{w(y)}{w(x_i)}l_i(y) = \sum_{i=1}^{n} w(x_i)r(x_i)t_i(y). \tag{3.14}$$

Using $x_i \notin (J_k)$, (3.13) and (3.14) yield

$$|w(y)r(y)| \leq c|w(y)r(y)|\frac{n^{7/6}T_n^{1/6}\delta_n}{3^{\varrho}}\lambda_n(w, y),$$

whence as $w(y)r(y) \neq 0$, we get (3.11) with a constant $c_2 > 0$, actually for every $0 < \delta_n \leq 1/2$ (say).

**3.4.** Let us apply Lemma 3.3 for every long interval $J_k$ with $q_k = 1/\log n$, say. By (3.12), we get the relation $\varrho(k,n) > n\delta_n/\log^2 n \gg n^{2/3}$, whence by (3.11) and $1 < T_n = o(n^2)$

$$\lambda_n(w,x) \gg n, \qquad x \in D_{1n} \setminus H_{1n}, \tag{3.15}$$

where $D_{1n} = \bigcup_k \{J_k : J_k \text{ is long}\}$ and $H_{1n} = \bigcup_k \{h_k : J_k \text{ is long}\}$. By construction

$$|H_{1n}| \le \sum |h_k| \le 4\sum q_k|J_k| \le \frac{4}{\log n}a_n, \tag{3.16}$$

where the summations are over $k : J_k \subset D_{1n} \subset I_n$. That is (2.1) holds for the long intervals in $I_n$, apart from a set of measure $\le 4a_n/\log n$. If $|K_n| = 0$, the same argument works for the whole interval $J_{kn} = I_n$.

**3.5.** Next, we consider the short intervals (subsections 3.5–3.9). Let $\varphi_n$ denote the number of short intervals $J_{kn}$, $1 \le k \le N-1$. If $\varphi_n \le n^\gamma$, then their total measure $\le n^\gamma a_n\delta_n = o(a_n)$, whenever $0 < \gamma < 1/6$, which we suppose from now on. So adding them to the exceptional set $H_n$, we get, using (3.16) and (3.11),

$$|H_n| \le |H_{1n}| + o(a_n) + 2a_n\delta_n + 2(a_n - b_n) < \varepsilon a_n$$

that is we would get the theorem (the third term, $2a_n\delta_n$, estimates the measure of the (possibly) short interval(s) $J_{Nn}$ and (or) $J_{0n}$; the fourth one measures the set $[-a_n, a_n] \setminus I_n$).

**3.6.** So from now on we can suppose $\varphi_n > n^\gamma$. First we introduce some further notations. With $\Omega_n(x) = \omega_n(x)w(x)$, let $u_k = u_k(q_k)$ be defined by

$$|\Omega_n(u_k)| := \min_{x\in J_k(q_k)} |\Omega_n(x)|, \qquad 1 \le k \le N-1,$$

( $|\Omega_n(u_k)| > 0$, as $q_k > 0$). Further let

$$|J_i, J_k| := \max(|z_{i+1} - z_k|, |z_{k+1} - z_i|), \quad 1 \le i, k \le N-1,$$
$$\varrho(J_i, J_k) := \min(|z_{i+1} - z_k|, |z_{k+1} - z_i|), \quad 1 \le i, k \le N-1.$$

We prove (see [15, Lemma 3.4 and its references]) the following lemma.

LEMMA 3.4. *Let $1 \le k, r \le N-1$. Then if $w \in \mathscr{E}(\mathbb{R})$,*

$$|t_k(x)| + |t_{k+1}(x)| > \frac{1}{4}\frac{|\Omega_n(u_r)|}{|\Omega_n(u_k)|}\frac{|\overline{J}_k|}{|J_r, J_k|}, \quad n \ge 2, \tag{3.17}$$

*whenever $x \in J_r(q_r)$, $\varrho(J_r, J_k) \ge a_n\delta_n$ and $|J_r| \le a_n\delta_n$. Here $t_k$ and $t_{k+1}$ are the fundamental functions corresponding to $z_k$ and $z_{k+1}$, respectively.*

PROOF. The proof of this lemma is similar to the one in [15]. We include it for sake of completeness. First we verify relation

$$|t_s(x)| = \left|\frac{\Omega(x)}{\Omega'(z_s)(x-z_s)}\right| = \frac{|\Omega(x)|}{|\Omega(u_r)|}\left|\frac{u_r - z_s}{x - z_s}\right||t_s(u_r)|$$
$$\geq \frac{1}{2}|t_s(u_r)| \quad \text{if } s = k,\ k+1 \text{ and } x \in J_r(q_r). \tag{3.18}$$

Indeed,

$$\frac{|u_r - z_s|}{|x - z_s|} \geq \frac{\{|u_r - z_s| + a_n\delta_n\} - a_n\delta_n}{|u_r - z_s| + a_n\delta_n} \geq 1 - \frac{a_n\delta_n}{2a_n\delta_n} = \frac{1}{2},$$

which gives (3.18). So we can write if $r < k$, say,

$$|t_k(x)| + |t_{k+1}(x)| \geq \frac{1}{2}\{|t_k(u_r)| + |t_{k+1}(u_r)|\}$$
$$= \frac{1}{2}\left|\frac{\Omega(u_r)}{\Omega(u_k)}\right|\left\{|t_k(u_k)|\frac{z_k - u_k}{u_r - z_k} + |t_{k+1}(u_k)|\frac{u_k - z_{k+1}}{u_r - z_{k+1}}\right\}$$
$$\geq \frac{1}{2}\frac{|\Omega(u_r)|}{|\Omega(u_k)|}\frac{q_k|J_k|}{|J_r,\ J_k|}\{|t_k(u_k)| + |t_{k+1}(u_k)|\}, \quad x \in J_r(q_r). \tag{3.19}$$

To obtain (3.17), we use [7, Theorem 1] which is stated as follows.

STATEMENT 3.5. Let $(a, b) \subseteq \mathbb{R}$ and $w = e^{-Q}\colon (a, b) \to (0, \infty)$. Assume that $Q'$ exists and is non-decreasing in $(a, b)$. Then for $1 \leq k \leq n-1$

$$|t_{kn}(w, X, x)| + |t_{k+1,n}(w, X, x)| \geq 1 \qquad \text{if } x \in [x_{k+1,n}, x_{kn}] \tag{3.20}$$

for arbitrary interpolatory $X \subset (a, b)$.

Applying (3.20) we obtain (3.17), considering that $2q_k|J_k| = |\overline{J}_k|$.

REMARKS. (1) Actually, if $x \in [x_{k+1}, x_k]$, then $t_s(x) \geq 0$ ($s = k, k+1$).
(2) Relation (3.20) is a generalization of an old theorem of Erdős and Turán which says that for an arbitrary interpolatory $X$,

$$l_{kn}(X, x) + l_{k+1,n}(X, x) \geq 1 \quad \text{if} \quad x \in [x_{k+1,n}, x_{kn}],\ 1 \leq k \leq n-1$$

(see [3; Lemma 4, p. 529]).

**3.7.** The following statement gives a result of Vértesi [14, Lemma 3.3] in a slightly different form.

STATEMENT 3.6. Let $F_k = [A_k, B_k]$, $1 \le k \le t$, $t \ge 2$ be any $t$ intervals in $[-A, A]$ with $|F_k \cap F_j| = 0$ $(k \ne j)$, $|F_k| \le A\delta$ $(1 \le k,\ j \le t)$, $\sum_{k=1}^{t} |\overline{F}_k| = A\mu$. Let $\xi \ge \delta$. If with a fixed integer $R \ge 4$ we have $\mu \ge 2^R\xi$, then there exists the index $s$ $(1 \le s \le t)$ such that

$$S:\ = \sum_{\substack{k=1 \\ \varrho(F_s, F_k) \ge A\xi}}^{t} \frac{|\overline{F}_k|}{|F_s, F_k|} \ge \frac{R\mu}{8} - \frac{3}{2}. \tag{3.21}$$

$F_s$ will be called the accumulation interval of $\{F_k\}_{k=1}^t$.

Here the definitions of $\overline{F}_k = \overline{F_k(q_k)}$, $|F_s, F_k|$ and $\varrho(F_s, F_k)$ correspond to the previous ones; $\mu$, $\delta$ and $\xi$ are fixed positive real numbers.

**3.8.** Now we define $q_k$ for the short intervals. Let $D_{2n}:\ = \bigcup_{k=1}^{n-1}\{J_k : |J_k| \le a_n\delta_n\}$ and $K_{2n}:\ = \{k : |J_k| \le a_n\delta_n,\ 1 \le k \le N-1\}$, $|K_{2n}| = \varphi_n$. If $m_k$ denotes the middle point of $J_k$, let

$$\begin{aligned}
\beta_{kn}:\ &= \max\{y : z_{k+1} \le y \le m_k \text{ and (2.1) does not hold for } y\},\\
\gamma_{kn}:\ &= \min\{y : m_k \le y \le z_k \text{ and (2.1) does not hold for } y\},\\
d_{kn}:\ &= \max(\beta_k - z_{k+1}, z_k - \gamma_k),
\end{aligned}$$

finally

$$q_{kn} = q(J_{kn}) = d_{kn}/|J_{kn}|, \qquad k \in K_{2n}. \tag{3.22}$$

Using $\lambda_n(w, x_k) = 1$,we obtain that $q_k > 0$. Further by definition, (2.1) holds true whenever $x$ is from the interior of $J_k(q_k)$, $k \in K_{2n}$. For the remaining 'bad' sets $\overline{J}_k$ we prove relation

$$\sum_{k \in K_{2n}} |\overline{J}_k|:\ = a_n\mu_n \le \frac{a_n\varepsilon}{2} \qquad \text{if} \qquad n \ge n_1. \tag{3.23}$$

Clearly, we can suppose that $n \in \{n_i\} = N_1$ for which $\mu_n > \varepsilon/2$. Now we can apply Statement 3.6 with the cast $\{F_r\} = \{J_{kn}\}_{k \in K_{2n}} = D_{2n}$, $A = a_n$, $\xi = \delta = \delta_n$, $\mu = \mu_n$, $R = [\log_2 n^{1/7}]$ and $n \in N_1$.

We get the accumulation interval and we denote it by $M_1 = M_{1n}$ (1st step). Dropping $M_{1n}$ we apply Statement 3.6 again, for the intervals $\{F_r\} = D_{2n} \setminus M_{1n}$

with $\mu = \mu_n - |\overline{M}_{1n}|/a_n \geq \mu_n - \delta_n > \mu_n/2$ and with the same $A, \xi, \delta, R$ and $N_1$. We get the accumulation interval $M_{2n}$ (2nd step). At the $i$th step ($3 \leq i \leq \psi_n$) we drop $M_{1n}, M_{2n}, \ldots, M_{i-1,n}$ and apply Statement 3.6 again for the intervals $\{F_r\} = D_{2n} \setminus \bigcup_{t=1}^{i-1} M_{tn}$ with $\mu = \mu_n - \sum_{t=1}^{i-1} |\overline{M}_{tn}|/a_n$ and with the same $A, \xi, \delta, R$ and $N_1$. Here $\psi_n$ denotes the first index for which

$$\sum_{t=1}^{\psi_n - 1} |\overline{M}_t| \leq \frac{a_n \mu_n}{2} \quad \text{but} \quad \sum_{t=1}^{\psi_n} |\overline{M}_t| > \frac{a_n \mu_n}{2}, \qquad n \in N_1. \tag{3.24}$$

Denoting by $M_{\psi_n+1,n}, M_{\psi_n+2,n}, \ldots, M_{\varphi_n,n}$ the remaining (that is not accumulation) intervals of $D_{2n}$, from relation (3.21) we get, if $n_1$ is big enough,

$$\sum_{k=r}^{\varphi_n}{}' \frac{|\overline{M}_k|}{|M_r, M_k|} \geq \frac{\mu_n \log n}{2 \cdot 7 \cdot 8} - \frac{3}{2} > \frac{\mu_n \log n}{120}, \qquad 1 \leq r \leq \psi_n, \qquad n \in N. \tag{3.25}$$

Here and later the dash on the summation indicates that we omit those indices $k$ for which $\varrho(M_r, M_k) < a_n \delta_n$.

**3.9.** By (3.22), we can choose the 'bad' points $v_{in} \in M_{in}(q_{in}/2)$ such that (2.1) does not hold for $v_{in}$ ($1 \leq i \leq \varphi_n, n \in N_1, q_{in} = q_{in}(M_{in})$).

If for a fixed $n \in N_1$ there exists an index $t$ ($1 \leq t \leq \varphi_n$) such that

$$\lambda_n(w, v_{tn}) \geq 2\, c\, \mu_n \log n \tag{3.26}$$

(where $c > 0$ will be determined later), then, using (2.1), we get relation $c\, \varepsilon \log n \geq \lambda_n(w, v_{tn})$, whence by (3.26), $2\mu_n \leq \varepsilon$. That means, we obtained (3.23). *We shall verify* (3.26) *for every fixed $n \in N_1$* with a proper $t = t(n)$. Indeed, otherwise for a certain $m \in N_1$

(3.27)
$$\lambda_m(w, v_{rm}) < 2c\, \mu_n \log m, \quad v_{rm} \in M_{rm}(q_{rm}/2), \quad \text{for every } r, \quad 1 \leq r \leq \varphi_m.$$

Then, by (3.27) and (3.23)

$$\sum_{r=1}^{\varphi_m} |\overline{M}_{rm}| \lambda_m(w, v_{rm}) < 2\, c\, a_m \mu_m^2 \log m. \tag{3.28}$$

On the other hand, applying (3.17) with $q_{kn}(M_{kn})/2$ we can write (with the same $|\overline{M}_i|$, as above)

$$\begin{aligned} |\overline{M}_r| \sum_{k=1}^{n} |t_k(v_{rn})| &\geq \frac{1}{2} |\overline{M}_r| \sum_{k \in K_{2n}}{}' \{|t_k(v_{rn})| + |t_{k+1}(v_{rn})|\} \\ &> \frac{1}{16} |\overline{M}_r| \sum_{k=1}^{\varphi_n}{}' \frac{|\Omega(\overline{u}_r)|}{|\Omega(\overline{u}_k)|} \frac{|\overline{M}_k|}{|M_r, M_k|}, \quad 1 \leq r \leq \varphi_n, \end{aligned}$$

for *arbitrary* $n \in N_1$ (here $|\Omega(\overline{u}_i)| = \min_{x \in M_i(q_i/2)} |\Omega(x)|$). Then, using relation $a + a^{-1} \geq 2$, (3.24) and (3.25), we get for $n \in N_1$

$$\begin{aligned}\sum_{r=1}^{\varphi_n} |\overline{M}_r| \lambda_n(w, v_{rn}) &> \frac{1}{16} \sum_{r=1}^{\varphi_n} \sum_{k=1}^{\varphi_n}{}' \frac{|\Omega(\overline{u}_r)|}{|\Omega(\overline{u}_k)|} \frac{|\overline{M}_r||\overline{M}_k|}{|M_r, M_k|} \\ &= \frac{1}{16} \sum_{r=1}^{\varphi_n} \sum_{k=r}^{\varphi_n}{}' \left\{ \frac{|\Omega(\overline{u}_r)|}{|\Omega(\overline{u}_k)|} + \frac{|\Omega(\overline{u}_k)|}{|\Omega(\overline{u}_r)|} \right\} \frac{|\overline{M}_r||\overline{M}_k|}{|M_r, M_k|} \\ &\geq \frac{1}{8} \sum_{r=1}^{\psi_n} |\overline{M}_r| \sum_{k=r}^{\varphi_n}{}' \frac{|\overline{M}_k|}{|M_r, M_k|} > \frac{a_n \mu_n^2 \log n}{8 \cdot 2 \cdot 120} \\ &= 2\, c\, a_n \mu_n^2 \log n \qquad \text{if } c = 1/3840.\end{aligned}$$

But this contradicts (3.28), that is (3.26) must hold for any $n \in N_1$ with a proper $t = t(n)$. So (3.23) has been proved.

**3.10.** Finally, we estimate $H_n$. If $J_{0n}$ is short, it should belong to $H_n$; the same holds for $J_{Nn}$. So by (3.16) and (3.23) (see subsection 3.5)

$$|H_n| \leq 4 \frac{a_n}{\log n} + \frac{a_n \varepsilon}{2} + 2 a_n \delta_n + 2(a_n - b_n) \leq \varepsilon a_n$$

which gives the theorem if $n \geq n_1(\varepsilon)$.

**3.11.** Proof of Theorem 2.2. The proof is analogous to the previous one after establishing the corresponding formula, so we only sketch it (subsections 3.11–3.14).

**3.12.** First let $w \in W$. The fact is that we have the same relations as before (for example, again $y_{kn}(w^2) - y_{k+1,n}(w^2) \sim a_n/n$, $y_{kn} \in I_n$), but of course, now $I_n$, $y_{kn}(w^2)$, $a_n(w)$, and so on, are defined for $w \in W$.

To be more precise, let $I_n = [-b_n, b_n]$ where, with $0 < \varepsilon < 1$, $b_n = a_n(1 - \varepsilon/5)$. As we know $a_n \to 1$ (see [4, p. 30, (ii)], say).

Relations corresponding to Statement 3.1 are [4, (1.35); p. 130, last row; (12.7) and (1.39)] respectively. Notice that we used relations $a_n \sim 1$, $|y_{kn}| \leq b_n = a_n(1 - \varepsilon/5)$, $\delta_n :\, = (nT_n)^{-2/3} = o(1)$ (see [4, (1.23)]), $\Psi_n(x) \sim \Phi_n(x) \sim 1$, if $x \in I_n$ ([4, (11.11) and (11.10)]).

The relation corresponding to (3.6) can be proved as in the proof of Lemma 3.2: the relation corresponding to (3.7) is [4, (12.5)]; the corresponding Markov–Bernstein inequality is now [4, (12.16)].

Moreover, the definition of the class $W$ (see subsection 1.6) ensures that [7, Lemma] and [7, Theorem 1] hold true, whence, among others, Statement 3.5 can be applied.

Other details, which are based on the previously mentioned relations, can be left to the reader.

**3.13.** Let $w \in GJ$ be defined by formula (1.26), further let

$$I_n: = [-1,1] \setminus \bigcup_{r=0}^{m+1} \left( u_r - \frac{\varepsilon}{10(m+1)}, u_r + \frac{\varepsilon}{10(m+1)} \right)$$

(actually, $I_n$ does not depend on $n$, but for convenience, we keep this notation). Replacing $a_n$ by 1, the formulae corresponding to (3.1), (3.2) and (3.6) come from [10; Theorems 3.2 and 3.3].

Indeed, (3.1) is immediate from [10, (3.4)]. To get (3.2), first let us remark that in $I_n$, $w(n,x) \sim w(x) \sim 1$, where $w(n,x) = w_0(\sqrt{1-x} + 1/n) w_{m+1}(\sqrt{1+x} + 1/n) \prod_{r=1}^{m} w_r(|x-u_r| + 1/n)$. Now [10, (3.5)] yields formula (3.2), because for $\varphi(x) = \sin\vartheta$ $(x = \cos\vartheta)$, $\varphi(x) \sim 1$ if $x \in I_n$.

To get (3.6) (which is an improvement of (3.3)), we use [10, (3.6)] and the fact $w(x) \sim w(n,x) \sim 1$, $x \in I_n$, again.

Finally we verify

$$\|p_n(w^2)w\| \le c\sqrt{n} \tag{3.30}$$

(which corresponds to (3.4) if we replace $T_n$ by $n^2$). We use relation

$$\|Q_n(x)w(n,x)\| \sim \|Q_n(x)w(x)\| \tag{3.31}$$

valid for any $Q_n \in \mathscr{P}_n$ supposing that the weight $w$ satisfies the inequality

$$w(x) \le \frac{c}{|I|} \int_I w(x)dx, \tag{3.32}$$

for all intervals $I \subset [-1,1]$ and $x \in I$ where $c > 0$ is independent of $I$ and $x$ (see [9, (5.1) and (6.26)]).

However, if $w \in GJ$, then relation (1.28) involves (3.32), that means (3.31) holds true whenever $w \in GJ$. Then, if $y_j = y_{jn}(w^2)$ is the closest root to $x$ of $p_n(w^2,x)$ we can write

$$\begin{aligned} |p_n(w^2,x)w(n,x)| &\sim |p_n(w^2,x)w(n,y_j)| \\ &\sim |p_n'(w^2,y_j)w(n,y_j)||x-y_j| \\ &\le c\frac{n}{(\sin\vartheta_j)^{3/2}} \frac{\sin\vartheta_j}{n} \le c\sqrt{n}, \quad |x| \le 1, \end{aligned} \tag{3.33}$$

(see [10; (3.4)–(3.6)] moreover, relations $w(n,x) \sim w(n,y_j)$ and $|x-y_j| \le \sin\vartheta_j/n$), whence by (3.31) we get (3.30).

**3.14.** The above mentioned relations yield the analogue of Lemma 3.3 (again replacing $T_n$ by $n^2$). However to get the relation corresponding to (3.20) we cannot use Statement 3.5 because we do not have the conditions for $Q'$; we choose another way. By definition, $w(x) \sim 1$ whenever $x \in I_n$; so by the Erdős–Turán relation (see subsection 3.6, Remark 2) we can write

$$(3.34)\quad t_k(x) + t_{k+1}(x) = \frac{w(x)}{w(x_k)} l_k(x) + \frac{w(x)}{w(x_{k+1})} l_{k+1}(x) \geq c\,\{l_k(x) + l_{k+1}(x)\} \geq c,$$

if $x \in J_k \subset I_n$; here $c$ *does* depend on $\varepsilon$ and $w$. Other details in proving (2.2) when $w \in GJ$ are analogous to the previous ones, so they are left to the reader.

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