

AMENABILITY, l_p -HOMOLOGIES AND TRANSLATION INVARIANT FUNCTIONALS

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Abstract

We introduce a new homology theory for infinite graphs in order to generalize some results of Willis and Woodward on translation invariant functionals. We also extend some theorems of Gerl and Gromov.

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1. Introduction

Let G be a finite, connected, directed graph with vertex set $V(G)$ and edge set $E(G)$. A real function f on $V(G)$ is called a *boundary function* if there exists a function ω on $E(G)$, such that $\partial\omega = f$. That is, for each $p \in V(G)$, the sum of the values of ω on the edges pointing inward p minus the sum of the values of ω on the edges pointing outward p is equal to $f(p)$. One of the oldest results of algebraic topology is that the codimension of boundary functions on V is 1, or in other words, the zeroth homology space of a finite, connected graph is one dimensional. Now let G be an *infinite*, connected directed graph. The zeroth l_∞ -homology of the graph G is the factor space $H_0^\infty(G) = C_0^\infty(G)/B_0^\infty(G)$, where $C_0^\infty(G)$ is the space of all bounded real functions on $V(G)$, and $B_0^\infty(G)$ is the space of functions, which are boundaries of bounded functions on the edges. According to Block and Weinberger [1], the zeroth l_∞ -homology of a graph is non-zero if and only if the graph is amenable, where the amenability is defined using Følner sequences the same way as in the case of discrete groups. One might observe that by Poincaré-duality the above result implies the combinatorial version of Gromov's theorem on l_∞ -cohomology [4]. Namely, one

can prove that if X is an n -dimensional, oriented combinatorial manifold, then the n th simplicial l_∞ -cohomology of X is non-zero if and only if the dual graph of X is amenable. In Sections 3 and 4 we extend the above characterization of amenability to l_p -homologies and c_0 -homologies. The notion of homologies are related to the notion of translation invariant linear functionals. The l_p and c_0 function space of discrete groups were studied by Willis [7, 8] and Woodward [9]. They proved that for a finitely generated group Γ , $l_p(\Gamma)$ ($p > 1$) respectively $c_0(\Gamma)$ have non-zero translation invariant linear functionals if and only if Γ is amenable. In Section 5 we show that the results of Willis and Woodward can be extended for arbitrary infinite graphs of bounded vertex degrees. For an infinite graph G a translation is a bijection on $V(G)$, which moves the vertices by a bounded distance.

2. The construction of a binary scheme

In this section we show what is behind the proof of [1, Theorem 3.1]. We construct a *binary scheme* for non-amenable graphs in order to facilitate the solution of certain equations. First we consider the binary tree B . The vertex set of B ; $V(B)$ consists of the finite 0-1-sequences. The *root* vertex is the empty set \emptyset . It has two *children*, the vertex indexed by 0 and the vertex indexed by 1. In general, the vertex indexed by $\{i_1, i_2, \dots, i_k\}$ has two children: $\{i_1, i_2, \dots, i_k, 0\}$ and $\{i_1, i_2, \dots, i_k, 1\}$. Edges are drawn between two vertices if and only if one is the child of the other one. That is, each vertex in $V(B)$ has degree 3 except the root, which has degree 2. Now let G be a connected, undirected infinite graph with bounded vertex degrees. Its vertex set is equipped with the path-distance metric $d_G : V(G) \times V(G) \rightarrow \mathbb{N}$. We call such a graph a *nice graph*.

Let $U \subset V(G)$, then $B(U, k) = \{x \in V(G) : \min_{y \in U} d_G(x, y) \leq k\}$. The nice graph G is *amenable* [1], if there exists a sequence of finite sets $\{U_i\}_{i>0}$, such that

$$(1) \quad \lim_{n \rightarrow \infty} \frac{|B(U_n, 1)|}{|U_n|} = 1,$$

where $|U_n|$ denotes the cardinality of U_n . If G is the Cayley-graph of a finitely generated amenable group then $\{U_i\}_{i>0}$ can be chosen as an exhaustion of G [1]. Obviously, if G is non-amenable, then there exists a constant $l \in \mathbb{N}$ such that for any finite non-empty set $U \subset V(G)$:

$$(2) \quad \frac{|B(U, l)|}{|U|} \geq 2.$$

Let G be a nice graph. A *binary scheme* on G is a map $P : V(G) \times V(B) \rightarrow V(G)$ such that:

(i) for any $x \in V(G)$, the map $P(x, \cdot)$ is injective and $P(x, \emptyset) = x$, where \emptyset is the root vertex of the binary graph B ;

(ii) there exists a constant $m \in \mathbb{N}$ such that for any $x \in V(G)$ and adjacent vertices $a, b \in V(B)$,

$$d_G(P(x, a), P(x, b)) \leq m;$$

(iii) for any $y \in V(G)$, the inverse image of y , $P^{-1}(y)$, consists of at most two elements.

PROPOSITION 2.1. *If G is a nice, non-amenable graph, then there exists a binary scheme $P : V(G) \times V(B) \rightarrow V(G)$.*

PROOF. The proof is based on the following immediate corollary of [2, Theorem 4.1].

LEMMA 2.2. *There exist maps $\psi_1, \psi_2, \psi_3, \psi_4$ from $V(G)$ to $V(G)$, such that*

- (i) *if $\psi_i(x) = \psi_j(y)$, then $x = y$ and $i = j$;*
- (ii) *for some $n > 0$, $d(\psi_i(x), x) < n$ uniformly on $\{1, 2, 3, 4\} \times V(G)$.*

□

Now we define the map P by an inductive process using the ψ_i 's of Lemma 2.2. Let a_0, a_1, a_2, \dots be an enumeration of $V(G)$.

The first step is:

$$P(a_0, \emptyset) = a_0, P(a_0, \{0\}) = \psi_3(a_0), P(a_0, \{1\}) = \psi_4(a_0).$$

The second step is as follows:

$$\begin{aligned} P(a_0, \{0, 0\}) &= \psi_1(P(a_0, \{0\})), P(a_0, \{0, 1\}) = \psi_2(P(a_0, \{0\})) \\ P(P(a_0, \{0\}), \emptyset) &= P(a_0, \{0\}), P(P(a_0, \{0\}), \{0\}) = \psi_3(P(a_0, \{0\})), \\ P(P(a_0, \{0\}), \{1\}) &= \psi_4(P(a_0, \{0\})) \\ P(a_0, \{1, 0\}) &= \psi_1(P(a_0, \{1\})), P(a_0, \{1, 1\}) = \psi_2(P(a_0, \{1\})) \\ P(P(a_0, \{1\}), \emptyset) &= P(a_0, \{1\}), P(P(a_0, \{1\}), \{0\}) = \psi_3(P(a_0, \{1\})), \\ P(P(a_0, \{1\}), \{1\}) &= \psi_4(P(a_0, \{1\})). \end{aligned}$$

Still during the second step, we choose the first element a_i in the sequence a_0, a_1, \dots which is not yet in the image of P and define: $P(a_i, \emptyset) = a_i$, $P(a_i, \{0\}) = \psi_3(a_i)$, $P(a_i, \{1\}) = \psi_4(a_i)$.

For the k -step, let us consider the set $\{y_1, y_2, \dots, y_s\} \subset V(G)$ with the following properties:

(i) For any $1 \leq i \leq s$, there exists $x_i \times b_i \in V(G) \times V(B)$, such that $P(x_i, b_i) = y_i$ has already been defined in the first $k - 1$ steps.

(ii) On the other hand, $P(x_i, c_i)$ and $P(x_i, d_i)$ are still undefined, where c_i, d_i are the children of b_i .

Note that for any y_i , such a pair (x_i, b_i) is unique by our construction. Then for each y_i , we define:

$$\begin{aligned} P(x_i, c_i) &= \psi_1(y_i), P(x_i, d_i) = \psi_2(y_i), \\ P(y_i, \emptyset) &= y_i, P(y_i, \{0\}) = \psi_3(y_i), P(y_i, \{1\}) = \psi_4(y_i). \end{aligned}$$

Now we choose again the first a_j in the enumeration of $V(G)$, which is not yet defined to be in the image of P , and let

$$P(a_j, \emptyset) = a_j, P(a_j, \{0\}) = \psi_3(a_j), P(a_j, \{1\}) = \psi_4(a_j).$$

Inductively we can define P for all $(x, b) \in V(G) \times V(B)$.

3. Homology theories

The notion of uniformly bounded homology was introduced by Block and Weinberger [1]. We need an analogue of their construction. Let G be a nice graph. The vertex set $V(G)$ is a metric space and one can equip $V(G)^{i+1}$, the $(i+1)$ -fold Cartesian product of $V(G)$, with a metric as well:

$$d((x_0, x_1, \dots, x_i), (y_0, y_1, \dots, y_i)) = \max_{0 \leq j \leq i} d(x_j, y_j).$$

Denote by Δ_{i+1} the multidagonal in $V(G)^{i+1}$. For $1 < p < \infty$, let $C_i^{l_p}(G)$ be the vector space of infinite formal sums $c = \sum a_{\bar{x}} \bar{x}$, where $\bar{x} \in V(G)^{i+1}$, $a_{\bar{x}} \in \mathbb{R}$ satisfying the following two conditions:

- (i) $\sum |a_{\bar{x}}|^p < \infty$;
- (ii) there exists $R > 0$ (depending only on c) such that $a_{\bar{x}} = 0$, if $d(\bar{x}, \Delta_{i+1}) > R$.

The usual boundary operator $\partial : C_i^{l_p}(G) \rightarrow C_{i-1}^{l_p}(G)$ is defined by

$$\partial(x_0, x_1, \dots, x_i) = \sum_{j=0}^i (-1)^j (x_0, x_1, \dots, \hat{x}_j, \dots, x_i).$$

Then $\partial\partial = 0$, hence $(C_*^{l_p}(G), \partial)$ is a chain complex. The homology of this complex is the l_p -homology of G , $H_i^{l_p}(G)$. The c_0 -homology of G : $H_i^{c_0}(G)$ can be defined analogously. We only need to replace condition (1) with the following one:

- (1a) $a_{\bar{x}} \rightarrow 0$ as $d_G(x_0, q) \rightarrow \infty$, where q is a fixed vertex of G .

If H is another nice graph and $f : V(G) \rightarrow V(H)$ is a quasi-isometry in the sense of Gromov, then we can define $f_* : C_i^{l_p}(G) \rightarrow C_i^{l_p}(H)$ and $f_* : C_i^{c_0}(G) \rightarrow C_i^{c_0}(H)$ by

$$f_* \left(\sum a_{\bar{x}} \bar{x} \right) = \sum a_{\bar{x}} f(\bar{x}).$$

Following the argument of [1], one can see that f_* induces an isomorphism between $H_i^{l_p}(G)$ and $H_i^{l_p}(H)$, and between $H_i^{c_0}(G)$ and $H_i^{c_0}(H)$.

A very important observation of Block and Weinberger is that the 0th uniformly finite homology of G is zero if and only if G is non-amenable. Analogously, we have the following theorem.

THEOREM 1. *For a nice graph G , the following conditions are equivalent.*

- (i) G is non-amenable;
- (ii) $H_0^{l_p}(G) = 0$;
- (iii) $H_0^{c_0}(G) = 0$.

PROOF. Let G be a nice non-amenable graph. Then by Proposition 2.1 we have a binary scheme $P : V(G) \times V(B) \rightarrow V(G)$. Let δ_0 be the characteristic function of the root vertex B . Then we consider the following 1-chain $\Psi = \sum a_{x,y}(x, y)$ on B :

$$a_{\emptyset, \{0\}} = -\frac{1}{2}, a_{\emptyset, \{1\}} = -\frac{1}{2}, a_{\{0\}, \{01\}} = -\frac{1}{4}, \dots, a_{\{i_1, i_2, \dots, i_{k-1}\}, \{i_1, i_2, \dots, i_{k-1}, i_k\}} = -\frac{1}{2^k},$$

all other coefficients $a_{x,y}$ are zeroes. Then of course, $\partial\Psi = \delta_0$.

Now let f be a function on $V(G)$. For any $x \in V(G)$, we define $\Phi_x = P(x, \cdot)_*(\Psi)$. Then $\partial\Phi_x = \delta_x$, the characteristic function of x . Hence one can define $\Phi^f = \sum_{x \in V(G)} f(x)\Phi_x$. It is easy to see that $\partial\Phi^f = f$. Moreover, if $f \in l_p(G)$, ($p > 1$) then $\Phi^f \in C_1^{l_p}(G)$; and if $f \in c_0(G)$ then $\Phi^f \in C_1^{c_0}(G)$. It is important to note that $\Phi^f \notin C_1^{l_1}(G)$.

The proof of the ‘only if’ part is a little bit more complicated than in the case of uniformly finite chains [1], but the idea is similar. Suppose that G is a nice amenable graph. There exists a sequence of finite disjoint sets U_1, U_2, \dots such that

$$(3) \quad \frac{|B(U_n, 1)|}{|U_n|} \leq 1 + 2^{-n}.$$

First we consider the l_p -case ($p > 1$). Let $L_n = n^{-2/p}|U_n|^{-1/p}$ and let δ_n be the characteristic function of U_n . Let $f = \sum_{n=1}^{\infty} L_n \delta_n$. Then

$$\sum_{x \in V(G)} |f(x)|^p = \sum_{n=1}^{\infty} |U_n| L_n^p = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

That is, $f \in l_p(G) = C_0^{l_p}(G)$. Suppose that $f = \partial\Phi$, where $\Phi = \sum a_{x,y}(x, y) \in C_1^{l_p}(G)$. Then

$$L_n|U_n| = \sum_{x \in U_n} \partial\Phi(x) = \sum_{x \in U_n} \sum_{y \in V(G)} (-a_{x,y} + a_{y,x}) = \sum_{x \in U_n} \sum_{y \notin U_n} (-a_{x,y} + a_{y,x}).$$

The numbers of pairs (x, y) such that $x \in U_n$, $y \notin U_n$ and $a_{x,y} \neq 0$, is smaller than $K 2^{-n}|U_n|$, where K is a positive real number depending only on G . Therefore by Jensen's inequality,

$$\sum_{x \in U_n} \sum_{y \notin U_n} (|a_{x,y}|^p + |a_{y,x}|^p) \geq \frac{(L_n|U_n|)^p}{(K 2^{-n}|U_n|)^{p-1}} = \frac{L_n^p|U_n|}{K^{p-1}(2^{-n})^{p-1}} = \frac{(2^n)^{p-1}}{n^2 K^{p-1}}.$$

Note that $(2^n)^{p-1}/(n^2 K^{p-1})$ tends to infinity as n tends to infinity, hence Φ cannot be in $C_1^{l_p}(G)$.

Now we handle the c_0 -case. Let $\{U_n\}_{n>0}$ and δ_n be as above then $f = \sum_n \delta_n/n \in C_0^{c_0}(G)$. Suppose that $f = \partial\Phi$, where $\Phi = \sum a_{x,y}(x, y) \in C_1^{c_0}(G)$. Then again we have the following estimate:

$$\frac{1}{n}|U_n| \leq \sum_{x \in U_n} \sum_{y \notin U_n} (|a_{x,y}| + |a_{y,x}|) \leq \sup_{x,y} |a_{x,y}| 2K 2^{-n} |U_n|.$$

Therefore $\Phi \notin C_1^{c_0}(G)$. □

(Note: A different proof for the l_p -case was given by Whyte [6].)

4. Simplicial homologies

Let G be a nice, directed graph. Now instead of $C_1^{l_p}(G)$, we consider the space of p -summable functions on the edges and denote it by $\tilde{C}_1^{l_p}(G)$. The simplicial boundary map $\tilde{\partial} : \tilde{C}_1^{l_p}(G) \rightarrow C_0^{l_p}(G)$ is defined as usual:

$$\tilde{\partial} f(x) = \sum_{e=a \rightarrow x} f(e) - \sum_{e=x \rightarrow b} f(e).$$

Note that $\tilde{C}_1^{l_p}(G) \subset C_1^{l_p}(G)$ and the boundary map on $\tilde{C}_1^{l_p}(G)$ is just the restriction of the boundary map on $C_1^{l_p}(G)$. The simplicial l_p -homology, respectively c_0 -homology is defined as

$$\begin{aligned} \tilde{H}_0^{l_p}(G) &= \tilde{C}_0^{l_p}(G) / \tilde{\partial} \tilde{C}_1^{l_p}(G), \\ \tilde{H}_0^{c_0}(G) &= \tilde{C}_0^{c_0}(G) / \tilde{\partial} \tilde{C}_1^{c_0}(G). \end{aligned}$$

A well-known result of Gerl [3] implies that $\tilde{H}_0^{l_2}(G)$ is zero if and only if G is non-amenable. We extend this implication.

THEOREM 2. *The following three conditions are equivalent.*

- (i) G is non-amenable.
- (ii) For $p > 1$, $\tilde{H}_0^{l_p}(G) = 0$.
- (iii) $\tilde{H}_0^{c_0}(G) = 0$.

PROOF. If G is amenable, then $\tilde{H}_0^{l_p}(G)$ and $\tilde{H}_0^{c_0}(G)$ are non-vanishing by Theorem 1. Now let G be a nice non-amenable graph. Let $f \in C_0^{l_p}(G)$ and $\Psi \in C_1^{l_p}(G)$ be such that $\partial\Psi = f$. We construct a $\tilde{\Psi} \in \tilde{C}_1^{l_p}(G)$ such that $\tilde{\partial}\tilde{\Psi} = f$.

First of all, for each pair $(a, b) \in \text{supp}(\Psi)$, we choose a path $L_{a,b}$ of minimal length from a to b . It is easy to see that there exists a real constant $K > 0$, depending only on Ψ such that each edge of G is contained in at most K chosen paths. Now we construct a finitely supported function $\Psi(a, b)$ on the edges for each pair (a, b) , $a \neq b$. Let $\Psi_{a,b}(e) = \Psi(a, b)$ if e is in the path $L_{a,b}$ with its own orientation. On the other hand, let $\Psi_{a,b}(e) = -\Psi(a, b)$ if e is in the path $L_{a,b}$ with reverse orientation. Finally, let $\Psi_{a,b}(e) = 0$ if the path $L_{a,b}$ does not go through the edge e at all. Then $\Psi_{a,b}$ is supported on at most M edges, where M depends only on G , also $(\tilde{\partial}\Psi_{a,b})(b) = \Psi(a, b)$, $(\tilde{\partial}\Psi_{a,b})(a) = -\Psi(a, b)$, otherwise $\tilde{\partial}\Psi_{a,b} = 0$. By our previous observation, $\tilde{\Psi} = \sum_{(a,b) \in \text{supp} \Psi} \Psi_{a,b}$ is a well-defined element of $\tilde{C}_1^{l_p}(G)$ and $\tilde{\partial}\tilde{\Psi} = f$. The c_0 -case can be handled the same way. \square

As an immediate corollary of Theorem 2 we can obtain the simplicial analogue of Gromov's theorem on the vanishing of the n th l_∞ -cohomology of a non-compact n -dimensional Riemannian manifold.

Let K be an infinite, connected, oriented n -dimensional combinatorial manifold. This means that K is the union of oriented n -simplices such that any $(n-1)$ -simplex of K is the face of exactly two n -simplices with different orientations. Also, the dual graph of K is nice. Recall that the vertices of the dual graph are the n -simplices of K and two n -simplices are defined adjacent if they have a common $(n-1)$ -face. Then one can define the n -th simplicial l_p -cohomology of K , $H_{(p)}^n(K)$ as in [5].

PROPOSITION 4.1. *Let K be as above. Then $H_{(p)}^n(K) = 0$ if and only if the dual graph of K is non-amenable.*

5. Translation invariant linear functionals

Let G be a nice graph. We call a bijection $T : V(G) \rightarrow V(G)$ a *translation* if there exists a constant $L > 0$ such that $d_G(x, T(x)) < L$ for all $x \in V(G)$. The translations form a group $T(G)$. Note that if G is the Cayley graph of a finitely generated group Γ , then for any $\gamma \in \Gamma$, $T_\gamma : x \rightarrow x\gamma$ is a translation. Thus we have an injective homomorphism from Γ to $T(G)$. For all the Banach spaces we investigated, namely

$l_\infty(G)$, $l_p(G)$ and $c_0(G)$, the group $T(G)$ can be faithfully represented by the left regular action the following way. If $f \in l_\infty(G)$ then $(Tf)(x) = f(T^{-1}(x))$. Let $B \subset l_\infty(G)$ be a $T(G)$ -invariant subspace (for example, $l_p(G)$, $c_0(G)$), then we call $\Phi : B \rightarrow \mathbb{R}$ a *translation invariant linear functional* if it commutes with the $T(G)$ -action.

According to Willis [7, 8] and Woodward [9], if Γ is a finitely generated group, then there exist non-zero Γ -invariant linear functionals on $l_p(G)$ ($p > 1$) and $c_0(G)$ if and only if G is amenable. We prove the analogue of their results for arbitrary nice graphs.

THEOREM 3. *Let G be a nice amenable graph and let $\alpha : H_0^{l_p} \rightarrow \mathbb{R}$ be a linear map. Then $\alpha^* : f \rightarrow \alpha[f]$ is a translation invariant linear functional on $l_p(G)$. The analogous statement holds for $c_0(G)$ as well.*

PROOF. It is enough to prove that if T is a translation and $f \in l_p(G)$, then $Tf - f$ is always a 0-boundary. Let $\Psi \in C_1^{l_p}(G)$ be defined by $\Psi = \sum a_{x,y}(x, y)$, where $a_{x,y} = f(x)$ if $y = T^{-1}(x)$, otherwise $a_{x,y} = 0$. Then $Tf - f = \partial\Psi$. \square

Now we prove the complementary result for non-amenable graphs.

THEOREM 4. *Let G be a nice, non-amenable graph. Then any translation invariant linear functional on $l_p(G)$ and $c_0(G)$ is zero.*

We will use the following easy-to-prove ‘colouring’ lemma.

LEMMA 5.1. *Let H be a graph such that $k \geq \deg(x)$ for any $x \in V(H)$. Then there exists a function $\pi : E(H) \rightarrow \{1, 2, \dots, 2k - 1\}$ such that if e and f have a joint vertex, then $\pi(e) \neq \pi(f)$.*

PROOF (of Theorem 4). Suppose that $\alpha(f) \neq 0$, where α is a translation invariant linear functional on $l_p(G)$ and $f \in l_p(G)$. By Theorem 1, there exists $\Psi \in C_1^{l_p}(G)$, $\Psi = \sum a_{x,y}^\Psi(x, y)$ such that $\partial\Psi = f$. Choose a constant $c > 0$ such that $a_{x,y}^\Psi = 0$ if $d_G(x, y) > c$. Now let H be the graph, obtained from G in the following way: $V(H) = V(G)$ and $(x, y) \in E(H)$ if $d_G(x, y) \leq c$. Obviously H is nice. We denote the bound on its vertex degrees by b_H . Let $\pi : E(H) \rightarrow \{1, 2, \dots, 2b_H - 1\}$ as in the previous lemma. Now we can define the translations T_i for $1 \leq i \leq 2b_H - 1$. Let $T_i(x) = y$ if there exists an edge $(x, y) \in H$ such that $\pi(x, y) = i$ and let $T_i(x) = x$ otherwise. It is easy to check that all T_i ’s defined above are in fact translations. Also, if $0 < d_G(x, y) \leq c$ then there exists exactly one i , $1 \leq i \leq 2b_H - 1$, such that $T_i(x) = y$. We define $g_i \in l_p(G)$ as follows. Let $g_i(x) = \Psi(x, y)$ if there exists a

vertex y such that $(x, y) \in E(H)$ and $\pi((x, y)) = i$. Otherwise let $g_i(x) = \Psi(x, x)$. Then,

$$\sum_{i=1}^{2b_H-1} (T_i g_i - g_i)(x) = f(x),$$

since $f(x) = \sum_y (\Psi(y, x) - \Psi(x, y))$. Then

$$0 \neq \alpha(f) = \sum_{i=1}^{2b_H-1} \alpha(T_i g_i - g_i) = \sum_{i=1}^{2b_H-1} 0 = 0$$

is a contradiction. □

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